Geometric singular perturbation theory for reaction-diffusion systems

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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Abstract

Geometric singular perturbation theory (GSPT) has proven to be an invaluable tool for the study of multiple time scale ordinary differential equations (ODEs). The extension of the theory to partial differential equations (PDEs) presents significant challenges as it implies a passage from finite-dimensional to infinite-dimensional dynamics. As a first step in this direction, we restrict ourselves to reaction-diffusion systems and generalise several aspects of GSPT for ODEs to such systems. Firstly, we explore the non-hyperbolic case, where we treat dynamic fold and Hopf bifurcations of fast-slow PDE-PDE systems with slowly varying bifurcation parameter. Our approach is to work with the Galerkin discretisation and employ standard GSPT along with estimates controlling the contribution of higher order modes. The method is not limited to the studied examples and can be generalised to a large class of problems. Secondly, we treat a fast-slow PDE-ODE system exhibiting bifurcation delay by constructing slow manifolds and combine this with application of a center manifold theorem. This provides a framework for dealing with such problems and although the results are not new, they have not been applied to the particular case of multiple scale systems.

Lay summary

The indispensability of mathematical multiple scale theory across a large breadth of scientific fields cannot be overstated. Almost every physical process exhibits dynamics in varying scales of length, time or energy and may involve highly non-trivial interactions between quantities evolving on these different scales, necessitating the development of rigorous mathematical theories and methods that are able to qualitatively explain and predict the behaviour of such processes by exploiting the underlying multiple scale structure. Many multiple scale models also have a spatial dependence, that current approaches are not able to handle properly due to immense mathematical complexities this presents. This results to simplifying assumptions such as averaging over the spatial domain in order to tackle the spatial dependence. The contribution of this work is the development of novel methods that allow dealing with this complexity in the original, unsimplified problem.

To my grandparents, in loving memory.

Contents

Chapter 0

Introduction

The indispensability of mathematical multiple scale theory across a large breadth of scientific fields cannot be overstated. Almost every physical process exhibits dynamics in varying scales of length, time or energy and may involve highly non-trivial interactions between quantities evolving on these different scales, necessitating the development of rigorous mathematical theories and methods that are able to *qualitatively* explain and predict the behaviour of such processes by exploiting the underlying multiple scale structure.

We do not have to look far to find important examples. One of the most celebrated applications can be found in biology and particularly the transmission of signals between neurons. It has been experimentally observed that the action potential across the neuron changes much more quickly than the neurotransmitter concentration in the synaptic cleft. This multiple time scale structure can be seen for certain parameter regimes in the famous Hodgkin-Huxley model [28] and its simplified version, the FitzHugh-Nagumo model. Another example from electronics, is the family of bistable oscillators such as the van der Pol oscillator [47], that are frequently used to model electric circuits. Here, the multiple scale structure of the model is encoded by a small damping parameter that leads to behaviours such as relaxation oscillations, which multiple scale analysis can predict. Applications also include fluid dynamics, where in many situations the Reynolds number becomes a small or large perturbation parameter [44]. A complete enumeration of all the applications of multiple scale dynamics is a vast undertaking on its own and outside the scope of this thesis; we refer to books such as [37, 38] for more details.

Multiple scale dynamics

The main object of study of multiple scale dynamics, are ordinary differential equation systems of the form

$$z' = F(z, \varepsilon),$$

where $0 < \varepsilon \ll 1$ is a small perturbation parameter and $z \in \mathbb{R}^k$. The presence of a singular perturbation is indicated [21] by examining the set $\{z \in \mathbb{R}^k : F(z, 0) = 0\}$; if it contains non-isolated points then we have a singular perturbation. The first step to analyse such problems is to put them in the *standard form*

$$\varepsilon\dot{u} = f(u, v, \varepsilon), \tag{1a}$$
$$\dot{v} = g(u, v, \varepsilon) \tag{1b}$$

where $u, v \in \mathbb{R}^{m\times n}$ with $m + n = k$, which is always possible locally [21].

The mathematical literature on multiple scale dynamics and perturbation theory is vast and includes methods such as matched asymptotics [10, 42, 41, 48], non-standard analysis [11, 12], invariant manifold theory [22, 32, 46, 27], geometric desingularisation [34, 17] and various classes of numerical methods [16, 23]. A collection of some of these approaches, known as *geometric singular perturbation theory* (GSPT) [21], provides a highly intuitive and natural understanding of the global dynamics via the construction of invariant manifolds that allow for dimensionality reduction.

If we set $\varepsilon = 0$ in (1),

$$0 = f(u, v, 0), \tag{2a}$$
$$\dot{v} = g(u, v, 0) \tag{2b}$$

we obtain an algebraic-differential system of equations, the *reduced problem.* It becomes clear that the set $\mathscr{C}_0 := \{(u, v) : 0 = f(u, v, 0)\}$ plays a central role for the dynamics of (1); it is known as the critical manifold. We interpret (2) as a dynamical system on $\mathscr{C}_0$.

Switching to the fast time $t = \tau/\varepsilon$ in (1), we find the equivalent system

$$u' = f(u, v, \varepsilon), \tag{3a}$$
$$v' = \varepsilon g(u, v, \varepsilon) \tag{3b}$$

where now the prime $\square'$ denotes differentiation with respect to t. In the singular limit $\varepsilon \to 0$ of the fast system, we find

$$u' = f(u, v, 0), \tag{4a}$$
$$v' = 0, \tag{4b}$$

the *layer problem.* The set of steady states of the layer problem is exactly $\mathscr{C}_0$ and the slow variables v become non-dynamic parameters in this limit.

A point $p \in \mathscr{C}_0$ is called normally hyperbolic if the derivative $D_u f(u, v, 0)|_p$ does not have spectrum on the imaginary axis. The most important result of geometric singular perturbation theory is that around such p, $\mathscr{C}_0$ locally persists as an invariant manifold $\mathscr{C}_\varepsilon$ for $\varepsilon > 0$ small, with stability properties determined by said spectrum, that is $\mathscr{O}(\varepsilon)$ close to $\mathscr{C}_0$ in the Hausdorff distance. In addition, the dynamics on the perturbed slow manifold approach those of the reduced problem (2). The existence of $\mathscr{C}_\varepsilon$ allows a reduction of the dimension of the problem. Put more simply, normal hyperbolicity implies the dynamics can be decomposed into fast and slow parts that can be studied separately, thus reducing the singlular to a regular perturbation. This, along with the persistence of the unstable and stable manifolds of the slow manifold as well as the foliated nature of those are the content of Fenichel's theorems [21, 5, 37].

Generically, there will also exist non-normally hyperbolic points or subsets of $\mathscr{C}_0$. In contrast to the hyperbolic case, the fast and slow dynamics can no longer be decoupled as intricate and complicated interactions between the two regimes can occur. The most successful approach in resolving these singular subsets is the blowup method or geometric desingularisation [34, 35, 31, 36, 45] an idea originating in algebraic geometry. The singular point, or subset more generally, is replaced by a higher dimensional object; for example, a sphere replaces a point or a cylinder replaces a line. In the extended phase space, one then gains additional hyperbolicity to proceed with the analysis. Intuitively, the process can be likened to an infinite zoom in on the singular subset in order to resolve the interactions between the fast and slow parts. Combining these two regimes, it becomes possible to obtain global properties of the vector field by tracking orbits across the normally hyperbolic and non-hyperbolic parts of $\mathscr{C}_0$.

Fast-slow PDEs

A natural question is whether the theory can be generalised to fast-slow partial differential equations (PDEs). A starting point is to consider reaction-diffusion systems of the form

$$\partial_t u = Au + f(u,v,\varepsilon), \tag{5a}$$
$$\partial_t v = \varepsilon\left(Bv + g(u,v,\varepsilon)\right), \tag{5b}$$

for $u(x,t), v(x,t)$, where $x \in U \subset \mathbb{R}^n$ is a domain, A, B are differential operators, accompanied by suitable boundary conditions. Existence and regularity results on solutions of reaction-diffusion systems are plentiful, making them natural candidates for deeper study.

Apart from the inherent mathematical interest in such an extension, it will open the way to a multitude of applications across situations where the spatial dependence is neglected. For example in many reaction-diffusion models such as (5), one has to restrict their attention to travelling wave solutions only, thus reducing the PDE to an ODE system, allowing application of the standard GSPT toolbox on this simplified problem. Similar simplifications are seen, among others, in problems from fluid dynamics, where symmetry assumptions are imposed in order to reduce the PDEs that describe the flow to ODEs. Thus, a broader insight into GSPT for PDEs is desirable.

As expected, such a generalisation remains an open question and progress has been slow until recently, since it means passing from dynamics on a finite-dimensional space to an infinite-dimensional one – thus losing important properties of the dynamics, such as time reversibility. The theory is more developed in the case $B \equiv 0$ with (5) taking the form

$$\partial_t u = Au + f(u,v,\varepsilon), \tag{6a}$$
$$\partial_t v = \varepsilon g(u,v,\varepsilon), \tag{6b}$$

with phase space $X \times \mathbb{R}^n$, that is, the fast variable u evolves in an infinite-dimensional function space X and the slow variables are finite-dimensional. It is well-established that several concepts and techniques from ODE dynamics can be lifted to problems of the type (5) and, in relation to GSPT, these include various stable and center manifold theorems as well as persistence results for invariant manifolds [26, 9, 7, 8, 5, 6]. This permits the extension of the normally hyperbolic case to fast-slow PDE-ODE systems such as (6) with a definition of normal hyperbolicity that can be extended in a straightforward way. Around points where this condition fails, one can employ the center manifold theorem for infinite-dimensional systems found in [24]. These two approaches are combined in chapter 3 to a simple toy problem to demonstrate the similarity we can achieve compared to dealing with purely ODE fast-slow systems. Despite the fact that none of the results given there are novel, the combination of a simple construction of slow manifolds along with the application of the center manifold theorem, as to provide a toolbox similar to classical GSPT has, to the best of my knowledge, not been attempted so far.

Returning to the full PDE-PDE system (5), with a non-trivial, unbounded, differential operator B, much less is known with the exception of specific equations, such as the Maxwell–Bloch equation [40]. Intuitively, the reason is the term $\varepsilon B v$ leads to severe difficulties as it gives rise to an unbounded perturbation of the corresponding semiflow for small $\varepsilon > 0$ compared to the limit $\varepsilon = 0$. All the aforementioned work in the previous paragraphs that provides us with the persistence of slow manifolds can no longer be used, as a bounded perturbation of the semiflow in an appropriate norm for $\varepsilon > 0$ is an essential underlying assumption. In addition, there are problems with the notion of normal hyperbolicity, as one has to be careful in the case were B has spectrum on the imaginary axis; e.g. if $B = \Delta$ with Neumann boundary conditions. Then, a definition using expansion and contraction rates along directions normal to the critical manifold, like [27] fails. Even worse, there are issues with the concept of fast and slow variables. To demonstrate this, let us consider the case $A = B = \Delta$ on the domain $[-a, a] \subset \mathbb{R}$. Expanding (5) with respect to the natural orthonormal basis $\{\lambda_k, e_k(x)\}$ of $L^2([-a, a])$ we find the infinite system of ODEs

$$\partial_t u_k = \lambda_k u_k + \langle f(u, v, \varepsilon), e_k(x)\rangle, \tag{7a}$$

$$\partial_t v_k = \varepsilon \left(\lambda_k v_k + \langle g(u, v, \varepsilon), e_k(x)\rangle\right), \tag{7b}$$

where $k = 1, 2, \ldots$. Since $|\lambda_k| \to \infty$ as $k \to \infty$, we observe that for any small, fixed $\varepsilon > 0$ there exists k_0 such that $\varepsilon|\lambda_k| > |\lambda_1|$ for $k \geq k_0$. In other words, the "slow" variables $\{v_k, k \geq k_0\}$ are no longer slow. Retracing our steps to the original PDE, this implies that the usual meaning of a slow variable v, evolving at a different time scale from the fast variable u, can break down, depending on the norms used.

Nevertheless, some partial progress has been made recently towards solving the normally hyperbolic case in [30], where slow manifolds $S_{\varepsilon,\zeta}$ are constructed locally around points that possess an appropriate notion of normal hyperbolicity. Note that a new parameter ζ is required. To explain the necessity of the new parameter, intuition from the discretised system (7) is used, separating the slow variables v_k into two parts: fixing a $k_0 \geq 1$, we group $\{v_k, k > k_0\}$ with the fast variables, leaving us with the k_0 slow variables $\{v_k, 1 \leq k \leq k_0\}$. In the PDE level, this is encoded by the parameter $\zeta \approx 1/k^2$ that determines a splitting the phase space Y of the v-variable into a direct sum of two subspaces, $Y = Y_F^\zeta \oplus Y_S^\zeta$ such that $\dim Y_S^\zeta \approx \lfloor \zeta^{-1/2} \rfloor < \infty$ and projecting onto them, obtaining two new variables v_F and v_S that are the fast and slow parts of v respectively. This splitting provides a spectral gap and allows the construction of $S_{\varepsilon,\zeta}$ as a graph over Y_S^{ζ} via a Lyapunov-Perron argument.

The relation between k_0 and ζ is such that, as k_0 increases ζ decreases and as $k_0 \to \infty$ we have $\zeta \to 0$. Ideally, to obtain true slow manifolds in the sense of Fenichel theory, we would want to fix $\varepsilon > 0$ and take $S_{\varepsilon,0}$ as our slow manifold. However, as hinted through (7), this is not possible due to the infinite-dimensional character of the problem, at least with the current approach. Despite this, $S_{\varepsilon,\zeta}$ possesses properties one would expect, most importantly local exponential attraction and dynamics on them are given by the reduced problem.

At this point, it is perhaps useful to compare slow manifolds with the related concept of inertial manifolds. An inertial manifold for a dissipative PDE system is a finite-dimensional invariant manifold that attracts all orbits exponentially fast as $t \to \infty$ [14]. On the other hand, slow manifolds provide information for finite time and are only locally invariant and with boundary, due to hyperbolicity breaking down. Orbits have to be tracked along such boundaries using different methods, such as the blowup method mentioned earlier.

Geometric analysis of singularities through discretisation

Since in infinite dimensions a direct generalisation of the geometric desingularisation is not available, the question remains on how to extend the slow manifolds $S_{\varepsilon,\zeta}$ around singularities in the phase space were hyperbolicity is lost. The approach followed here, is inspired once again from (7) and the method developed in chapter 1 and chapter 2 is the primary original contribution of this thesis. Observe that truncating the discretisation (7) at a fixed $k_0 \in \mathbb{N}$ gives a fast-slow system of ODEs with k_0 fast and k_0 slow variables. The projection of critical manifold of the PDE onto the subspace generated by the first k_0 eigenvalues $\{e_k(x)\}_{k=1}^{k=k_0}$ is a part of the critical manifold $\mathscr{S}_{0,k_0}$ of the truncated system. From standard geometric singular perturbation theory, normally hyperbolic parts of $\mathscr{S}_{0,k_0}$ perturb to slow manifolds $\mathscr{S}_{\varepsilon,k_0}$ for small $\varepsilon > 0$. It has been shown in [39] that $S_{\varepsilon,\zeta}$ can be closely approximated by $\mathscr{S}_{\varepsilon,k_0}$ in a suitable norm provided k_0 is large enough. Choosing such a k_0, we then go on to analyse the discretised system by propagating $\mathscr{S}_{\varepsilon,k_0}$ through the singularity using the usual tool of geometric desingularisation for ODEs. The main difficulty is that we are working with systems of arbitrary number of equations. On top of that, the critical manifolds of the descretised systems consist of more parts than simply $\mathscr{S}_{0,k_0}$ and their geometry can be complex, with normally hyperbolic regions separated by $k_0 - 1$-dimensional submanifolds. Thus, in order to ensure that the initial conditions we consider approach the projected from the PDE singularity before hitting other non-hyperbolic parts of the critical manifold we perform careful estimates using the variation of constants formula.

This method of resolving singularities in PDEs based on discretisation was first applied in [19], where the PDE version of the dynamic pitchfork bifurcation with a slowly varying parameter is analysed. There, the spatial domain is the interval $[-a, a]$ for an $a > 0$ and attempting to apply the technique of geometric desingularisation necessitates appending the dummy equation $a' = 0$ to the system. This leads, after applying the blowup transformation, to vector fields that are not defined in the singular limit $a \to 0$. Apart from the conceptual issues this causes, it does not allow the application of the method to more general problems. For example, in the PDE version of the dynamic fold bifurcation we consider in chapter 1 this flavour of the method fails. Instead, here we consider an ε-dependent rescaling of the domain and then proceed to apply a suitable blowup transformation. This rescaling, combined with a-priori estimates based on the variation of constants formula can be used for a large class of problems. The flexibility of our approach is demonstrated in chapter 2, where we prove results analogous to [25] for the PDE version of the slow passage through Hopf bifurcation, using the exact same procedure.

The systems we work with in the first two chapters have been chosen due to the fact that the corresponding ODE problems have provided a blueprint on how to work with fast-slow problems in general. It is hoped that the resolution of singularities developed in the present will be part of a larger toolbox towards the goal of developing a version of geometric singular perturbation theory for fast-slow PDE systems.

Structure of this thesis

The thesis consists of three chapters. In chapter 1 we perform a geometric analysis of the planar fold [34] extended to a PDE system and is based on the preprint [18]; all parts of that preprint, with the exception of the part of section 1.4.3 on finite time blowup, were authored by me, using the invaluable input and comments of my co-authors. In chapter 2 we showcase the application of the same procedure to a PDE extension of a system that undergoes a bifurcation delay through a Hopf bifuraction [25].

The main contribution is the blowup analysis of part of the slow manifold of the discretised fast-slow PDE systems, which corresponds to the homogeneous in space solutions of (8). In more detail, both chapters deal with a system of the form

$$\begin{aligned} u_t &= u_{xx} + f(u,v,\varepsilon), \\ v_t &= \varepsilon(v_{xx} + g(u,v,\varepsilon)) \end{aligned} \tag{8}$$

on the domain $[-a,a]$, $a>0$ with zero Neumann boundary conditions. As the first eigenvalue of the Laplacian in this case is zero, the Galerkin discretisation truncated at a *fixed* k_0 reads

$$\begin{aligned} u_1' &= f(u_1,v_1,\varepsilon) + F_1(u_j,v_j,\varepsilon), \\ v_1' &= \varepsilon\left(g(u_1,v_1,\varepsilon) + G_1(u_j,v_j,\varepsilon)\right), \\ u_k' &= F_k(u_j,v_j,\varepsilon), \\ v_k' &= \varepsilon G_k(u_j,v_j,\varepsilon), \quad 1 \le j \le k_0, \end{aligned} \tag{9}$$

where the functions $F_k, G_k, 1 \le k \le k_0$ are determined from f, g. As we shall see, considering (9) on the subspace $\{u_j = v_j = 0, 2 \le j \le k_0\}$ reduces it to the fast-slow ODE

$$\begin{aligned} u_1' &= f(u_1,v_1,\varepsilon), \\ v_1' &= \varepsilon g(u_1,v_1,\varepsilon). \end{aligned} \tag{10}$$

In both systems we examine, this fast-slow ODE is well-understood. The full discrete system however features, as we shall see, much richer fast-slow structure, with k_0-dimensional critical manifolds and $k_0 - 1$-dimensional submanifolds along which normal hyperbolicity is lost. Thus, while restricting the system to the fist mode (that is, considering the dynamics on the invariant subspace $\{u_k = v_k = 0, 2 \le k \le k_0\}$) we obtain the normal form of a fold and Hopf bifuractions with slowing varying parameter in chapter 1 and chapter 2 respectively, considering the full phase space yields much more complex dynamics. Attempting to perform a complete fast-slow analysis would be very interesting due to the singular point of the restricted problem being part of larger manifolds of non-hyperbolic points, which does not permit using known GSPT results on standard forms of the various singularities. For our purposes however, we restrict to tracking the part of the critical manifold lying on the first mode within the full system, due to our interest on perturbations around the homogeneous in space initial values and the connection between the PDE and discretised system obtained in [39]. This creates the need of controlling the variables $u_j, v_j, 2 \le j \le k_0$ and hence the need for careful estimates arises.

It will become evident that the same procedure can be carried out with straightforward modifications to fast-slow PDEs resulting by adding a diffusion term to a standard form fast-slow ODE, reusing known results for the latter and complimenting it with the required estimates, which will vary for different choices of f, g but follow the same general approach. In addition, our ideas can be used to study perturbations and blowup around non-constant in space functions, something necessary when taking different boundary conditions, such as Dirichlet, or considering PDEs with different underlying solution spaces and expanding with respect to other bases.

The critical manifold S of (8) can be formally defined as the set of functions belonging in the appropriate function space X (will be made precise once we consider the specific problems) that satisfy the equation

$$S := \left\{(u, v) \in X \times X : u_{xx} + f(u, v, 0) = 0\right\}.$$

This is a nonlinear partial differential equation and in general can be difficult to determine the existence and number of solutions, thus making it difficult to work with the full critical manifold. Observe that when having Neumann boundary conditions, we can consider the subset $\mathscr{C}_0 \subset S$ of constant functions, given by

$$\mathscr{C}_0 := \left\{(u, v) \in \mathbb{R} \times \mathbb{R} : f(u, v, 0) = 0\right\}.$$

Taking intro account the previous discussion, $\mathscr{C}_0$ can be viewed as the critical manifold of (10) and in turn the part of the critical manifold of (9) that lies within the plane $\{u_k = v_k = 0, 2 \le k \le k_0\}$. Of course, S will contain more elements than just those of $\mathscr{C}_0$ and this can be seen on the discretised problem (9) in the form of larger critical manifold $\mathscr{C}$ containing $\mathscr{C}_0$. This full critical manifold $\mathscr{C}$ of (9) is the result of projecting S onto the corresponding k_0-dimensional subspace at each truncation level k_0. In each of chapter 1, chapter 2 examples of the form of $\mathscr{C}$ in the case of $k_0 = 2, 3$ can be found. When working with arbitrary k_0 however, we are only concerned with $\mathscr{C}_0$. In principle, for a fixed k_0, one could analyse $\mathscr{C}$. This involves blowing-up a $k_0 - 1$-dimensional non-hyperbolic submanifold which can be very difficult as we do not even have an explicit formula for $\mathscr{C}$ and the non-hyperbolic submanifold, therefore focusing on $\mathscr{C}_0$. As mentioned already, this restriction requires estimates on the variables $u_k, v_k, 2 \leq k \leq k_0$ in order to control the interactions of the flow of (9) with the non-hyperbolic submanifold we ignore by focusing on $\mathscr{C}_0$ which, as mentioned, is sufficient for our purposes.

Since we perform a blowup of the origin of (9), one may wonder why we should expect the blowup to work and desingularise this fully degenerate steady state, as we work with a $2k_0$-dimensional system. As it turns out, the blowup transformation allows enough hyperbolicity to perform the analysis, despite not fully resolving the degeneracy.

No discussion about extending the fold point analysis to a PDE context would be complete without mentioning canards [34]. It is not expected that any canard solutions appear in the PDE version of the fold singularity, as these require an extra parameter and can only be observed for an exponentially small in ε interval of this parameter. In the corresponding PDE system, non-constant in space initial values, even close to the homogeneous functions are likely to disturb this delicate phenomenon. A thorough investigation of canards in fast-slow PDE-PDE systems should be a topic for further future research.

Finally, chapter 3 concerns a PDE-ODE system, with an infinite-dimensional fast variable and *finite-dimensional* slow variable. The main objective there is to combine a slow manifold construction along with a center manifold reduction to demonstrate how such systems can be handled. In essence, in the finite-dimensional slow variable setting, instead of trying to discretise and blowup the resulting singularity, we can work directly with the full system and deal with non-hyperbolic points via a center manifold reduction.

Open problems & future research

As mentioned, geometric singular perturbation theory for PDEs has only recently began a systematic development. Some important results have already been established, while many crucial questions remain and should the studied in the future. The following questions stem directly from the results obtained herein and can be consider as natural directions of research:

- Extend the new theory to all of the commonly used types of boundary conditions, in order for it to be applicable to broader classes of problems. Currently, only Neumann boundary conditions have been examined, with critical manifolds consisting of spatially homogeneous functions. An important intermediate step is to consider Dirichlet boundary conditions and consequently critical manifolds involving non-homogeneous functions.

- Blowup the submanifolds that separate the normally hyperbolic parts in the truncated systems. The loss of normal hyperbolicity at a single point at the PDE level manifests as $k_0 - 1$ non-normally hyperbolic surface of the projected k_0-dimensional critical manifold. This means that the interactions between fast and slow dynamics for PDEs exhibit behaviour that has no ODE analogue, in the sense that standard results on, e.g. the planar fold, or the transcritical singularity [35] cannot be applied due to the non-hyperbolic point being part of a non-hyperbolic submanifold. A full understanding requires blowing-up these submanifolds instead of a single point on them, as is currently done.

- The ultimate long term goal is a deeper understanding between the relation of invariant manifolds that can be constructed for the discretised problem and those that exist on the PDE level. Essentially, this is a fundamental question on the double limit $\varepsilon \to 0, k_0 \to \infty$ and the current methods do not provide a satisfactory answer.

Chapter 1

Geometric analysis of fast-slow PDEs with fold singularities

1.1 Introduction

In this chapter, which is based on the preprint [18], we perform a fast-slow analysis of the system of differential equations

$$u_t = u_{xx} - v + u^2 + H^u(u, v, \varepsilon), \tag{1.1a}$$

$$v_t = \varepsilon\left(v_{xx} - 1 + H^v(u, v, \varepsilon)\right) \tag{1.1b}$$

on the domain $[-a, a]$ with zero Neumann boundary conditions, where H^v and H^v are higher-order terms which are specified below. The problem is a generalisation to a PDE of the classical planar fold [34] and will hopefully provide a blueprint for applying GSPT to fast-slow PDEs.

Remark 1.1. Note that locally well-defined (smooth) solutions for (1.1) can be obtained from classical theory on sectorial operators with reaction kinetics [26] and parabolic regularity [20].

The Galerkin discretisation approach we use here to resolve the singularity at the origin was first employed in [19], where the transcritical singularity is analysed. The key element to obtain a useful blowup transform was the inclusion of the domain length a as a variable. This leads to a negative weight in the transformation that limits the application of that approach to different problems, such as in (1.1). In particular, it does not allow the lifting of the analysis of the corresponding ODE to the discretised system and leads to vector fields in the different charts that are not defined at the origin. Here, we instead use a domain rescaling by the slow parameter ε to avoid negative powers. This rescaling has the advantage of leaving the first Galerkin mode unchanged, allowing us to reuse the corresponding classical ODE analyis.

If one restricts to spatially homogeneous initial data only, the dynamics of (1.1) reduce to those of the corresponding ODE for the singularly perturbed planar fold, a well-known prototypical fast-slow system that was studied in [34] via the geometric blow-up technique. Here, we investigate the dynamics of (1.1) in a *full* neighbourhood of the origin in an infinite-dimensional phase space, in the sense that we will consider initial data that are close to the spatially homogeneous solution in an appropriately chosen norm.

This work is divided into two parts. In the first part, we apply results from [30] that will provide us with slow manifolds which drive the (semi)flow of (1.1) within a neighbourhood of the origin in the chosen phase space, provided the initial data are suitably chosen. Using results from [39], these manifolds can be approximated by slow manifolds in a truncated – and thus *finite-dimensional* – Galerkin discretisation of (1.1). To avoid confusion, we henceforth refer to these finite-dimensional manifolds as *Galerkin manifolds.* The resulting approximation can be made arbitrarily precise provided an appropriate truncation level, denoted by $k_0 > 0$, is chosen.

In the second part, which is the main result of this work, we extend these Galerkin manifolds around the singularity at the origin in the truncated, $(2k_0+1)$-dimensional Galerkin discretisation via the blow-up technique. There are evident similarities between our analysis and classical GSPT, where Fenichel's Theorem [21] is combined with geometric desingularisation in the form of the blow-up technique. However, the high dimensionality of our Galerkin discretisation, in combination with the inherent spatial dependence of Equation (1.1), poses various technical challenges. Thus, a preparatory rescaling of the domain length is introduced to allow for the application of the blow-up technique; further, careful consideration of initial data in tandem with various estimates on the evolution of the modes in the Galerkin discretisation is required to ensure that solutions do not exhibit finite-time blowup before reaching the singularity at the origin. These challenges naturally arise from the more complex structure of critical manifolds for infinite-dimensional dynamical systems.

Our main results can hence be summarised as follows; precise statements will be given below.

- Equation (1.1) possesses a family of slow manifolds $S_{\varepsilon,\zeta}$ for small $\varepsilon > 0$ and an additional control parameter $\zeta > 0$. These can be approximated by Fenichel-type slow manifolds $\mathscr{C}_\varepsilon = \mathscr{C}_{\varepsilon,k_0}$ in the corresponding Galerkin discretisation truncated at $k_0 > 0$, provided k_0 is sufficiently large.

- For any $k_0 > 0$ fixed, the Galerkin manifolds $\mathscr{C}_{\varepsilon,k_0}$ are extended around the fold singularity at the origin in the Galerkin discretisation, which we show by combining the well-known fast-slow analysis of the singularly perturbed planar fold with *a priori* estimates that control higher-order modes.

In summary, our work thus contributes to the ongoing development of a geometric approach for the study of singularities in multiple-scale (systems of) PDEs; we exemplify the approach through our study of a generic co-dimension one singularity, the singularly perturbed fold, which can be expected to arise in a wide variety of applications.

1.2 Galerkin discretisation

The starting point for our analysis is the following singularly perturbed system of PDEs,

$$u_t = u_{xx} - v + u^2 + H^u(u, v, \varepsilon) \qquad \text{for } x \in (-a, a) \text{ and } t > 0, \tag{1.2a}$$

$$v_t = \varepsilon(v_{xx} - 1 + H^v(u,v,\varepsilon)) \qquad \text{for } x \in (-a,a) \text{ and } t>0, \tag{1.2b}$$
$$u_x(t,x) = 0 = v_x(t,x) \qquad \text{for } x = \mp a \text{ and } t>0, \tag{1.2c}$$
$$u(0,x) = u_0(x) \text{ and } v(0,x) = v_0(x) \qquad \text{for } x \in (-a,a), \tag{1.2d}$$

where $a > 0$ is fixed. In analogy with the canonical form for the singularly perturbed planar fold studied in [34], we refer to u and v as the fast and slow variables, respectively, in (1.2). The functions H^u and H^v are assumed to be smooth and of the form

$$H^u(u,v,\varepsilon) = \mathcal{O}(\varepsilon, uv, v^2, u^3) \quad \text{and} \tag{1.3a}$$
$$H^v(u,v,\varepsilon) = \mathcal{O}(v^2). \tag{1.3b}$$

We assume that the higher-order terms in the slow variable H^v are orthogonal in $L^2(-a,a)$ to the subspace of constant functions, which is not an essential restriction and is only required for technical reasons, as will become apparent in our solution estimates for the system of ODEs resulting from a Galerkin discretisation of (1.2); see Lemma 1.36. In practice, we restrict H^v so that $H^v(u,v,\varepsilon)$ has zero mean over $[-a,a]$ for any $u,v \in L^2(-a,a)$. Example of such nonlinearities include taking a smooth $\tilde{H}^v : \mathbb{R}^3 \to \mathbb{R}$ and setting $H^v(u,v,\varepsilon) = \tilde{H}^v(u,v,\varepsilon) - \frac{1}{2a}\int_{-a}^{a} \tilde{H}^v(u,v,\varepsilon)$.

We could also assume more general forms of H^v, for example $H^v(u,v,\varepsilon) = \mathcal{O}(u^2, uv, v^2, \varepsilon)$ in analogy with the planar fold with the caveat that we would have to impose further restrictions to the initial values for higher order modes u_k, $k \geq 2$.

More compactly, we can write (1.2) as

$$w_t = Aw + F(w), \quad \text{with } w(0) = w_0,$$

where $w = (u,v)^T$, $w_0 = (u_0, v_0)^T$, $F(w) = (-v + u^2 + H^u(u,v,\varepsilon), -\varepsilon + \varepsilon H^v(u,v,\varepsilon))^T$, and

$$Aw = \begin{pmatrix} u_{xx} & 0 \\ 0 & \varepsilon v_{xx} \end{pmatrix}, \quad \text{with } \mathcal{D}(A) = \{w \in H^2(-a,a)^2 \,:\, u_x = 0 = v_x \text{ at } x = \mp a\}.$$

We have that $F(w)$ is locally Lipschitz continuous on $Z^\alpha = \mathcal{D}(A^\alpha)$ for $1/4 < \alpha < 1$. Moreover, the operator A is sectorial and a generator of an analytic semigroup on $Z = L^2(-a,a)^2$. Thus, for $w_0 \in Z^\alpha$, there exists a unique local-in-time solution $w \in C([0,t_*); Z^\alpha) \cap C^1((0,t_*); Z)$, with $w(t) \in \mathcal{D}(A)$, to (1.2) for some $t_* > 0$; see e.g. [26]. The quadratic nonlinearity in (1.2) implies a potential finite-time blowup of solutions to (1.2); cf. e.g. [4]. However, simple estimates show that, for initial $u_0 < 0$ and $v_0 > 0$, a solution of (1.2) exists for $t > 0$ such that $u(t) \leq 0$ and $v(t) \geq 0$.

Before giving a precise statement of our results, we introduce the Galerkin discretisation of the system of PDEs in (1.2) with respect to the eigenbasis $\{e_k(x) : k = 1,2,\ldots\}$ of the Laplacian on $L^2(-a,a)$ with Neumann boundary conditions. Specifically, the relevant orthonormal basis and the corresponding eigenvalues are given by

$$e_{k+1}(x) = \sqrt{\frac{1}{a}}\cos\left(\frac{k\pi(x+a)}{2a}\right) \quad \text{and} \quad \lambda_{k+1} = -\frac{k^2\pi^2}{4a^2} \quad \text{for } k = 1,2,\ldots, \tag{1.4}$$

respectively, with $e_1(x) = \frac{1}{\sqrt{2a}}$ and $\lambda_1 = 0$. We define

$$b_k := -(k-1)^2\pi^2, \tag{1.5}$$

so that $\lambda_{k+1} = \frac{b_{k+1}}{4a^2}$.

Then, solutions of (1.2) can be expanded as

$$u(x,t) = \sum_{k=1}^{\infty} e_k(x)u_k(t) \quad \text{and} \quad v(x,t) = \sum_{k=1}^{\infty} e_k(x)v_k(t). \tag{1.6}$$

Substitution of (1.6) into (1.2) results in the infinite system of ODEs

$$u_k' = \lambda_k u_k - \langle v, e_k\rangle + \langle u^2, e_k\rangle + \langle H^u, e_k\rangle, \tag{1.7a}$$
$$v_k' = \varepsilon\left(\lambda_k v_k - \langle 1, e_k\rangle + \langle H^v, e_k\rangle\right) \tag{1.7b}$$

for $k = 1, 2, \ldots$, where

$$\langle \phi, \psi\rangle = \int_{-a}^{a} \phi(x)\psi(x)\,dx \quad \text{for } \phi, \psi \in L^2(-a,a).$$

Using the formulae in (1.4), we can then derive the following explicit form of (1.7):

Proposition 1.2. *The system in* (1.7)*, truncated at* $k_0 \in \mathbb{N}$*, reads*

$$u_1' = -v_1 + \frac{1}{\sqrt{2a}}\sum_{j=1}^{k_0} u_j^2 + H_1^u, \tag{1.8a}$$
$$v_1' = -\sqrt{2a}\varepsilon, \tag{1.8b}$$
$$u_k' = \frac{1}{4}a^{-2}b_k u_k - v_k + \frac{2}{\sqrt{2a}}u_1u_k + \frac{1}{\sqrt{a}}\sum_{i,j=2}^{k_0}\eta_{i,j}^k u_i u_j + H_k^u, \tag{1.8c}$$
$$v_k' = \varepsilon\frac{1}{4}a^{-2}b_k v_k + \varepsilon H_k^v \tag{1.8d}$$

for $2 \le k \le k_0$*, where* $0 \le \eta_{i,j}^k \le 1$ *is non-zero if and only if* $i + j - 2 = k - 1$ *or* $|i - j| = k - 1$*, and*

$$H_1^u = \mathcal{O}(\varepsilon, v_1^2, v_j^2, u_1v_1, u_jv_j, u_1u_j^2, u_iu_ju_l) \quad \textit{for } 2 \le i,j,l \le k_0, \tag{1.9}$$
$$H_k^u = \mathcal{O}(v_1v_i, v_iv_j, u_1v_i, u_iv_1, u_iv_j, u_1^2u_i, u_1u_iu_j, u_iu_ju_l) \quad \textit{for } 2 \le i,j,l \le k_0, \textit{ and} \tag{1.10}$$
$$H_k^v = \mathcal{O}(v_1v_i, v_iv_j) \quad \textit{for } 2 \le i,j \le k_0. \tag{1.11}$$

Remark 1.3. Our assumption that the higher-order terms H^v are orthogonal to the subspace of constant functions ensures that $H_1^v = 0$ in (1.8).

Proof. Because the basis $\{e_j\}_{j\ge1}$ is orthonormal, we have $\langle v, e_k\rangle = v_k$ for all $k \ge 1$. We observe that $\langle 1, e_1\rangle = \sqrt{2a}$ and $\langle 1, e_k\rangle = 0$ for any $k \ge 2$. Recalling that e_1 is a constant function, we find

$$\langle e_1e_j, e_k\rangle = e_1\langle e_j, e_k\rangle = (2a)^{-1/2}\delta_{j,k}$$

for all $j, k \geq 1$, where $\delta_{j,k}$ denotes the standard Kronecker delta. In addition, a calculation shows that

$$\langle e_i e_j, e_k \rangle := a^{-1/2} \eta^k_{i,j},$$

where $\eta^k_{i,j}$ is independent of a and given by

$$\eta^k_{i,j} = \int_0^1 \cos\big((i+j-2)\pi x\big) \cos((k-1)\pi x)\, dx + \int_0^1 \cos\big((i-j)\pi x\big) \cos((k-1)\pi x)\, dx.$$

It follows that $0 \leq \eta^k_{i,j} \leq 1$ is non-zero if and only if $i + j = k+1$ or $|i-j| = k-1$. In particular, $\langle e_k^2, e_k \rangle = 0$ for $2 \leq k \leq k_0$. Equipped with the above, we can now calculate the term $\langle u^2, e_k \rangle$ in (1.7). For $k = 1$, we have

$$\begin{aligned}\left\langle \left(\sum_{j=1}^{k_0} u_j e_j \right)^2, e_1 \right\rangle &= \sum_{i,j=1}^{k_0} u_i u_j \langle e_j e_i, e_1 \rangle = \sum_{i,j=1}^{k_0} u_i u_j e_1 \langle e_j, e_i \rangle \\ &= (2a)^{-1/2} \sum_{j=1}^{k_0} u_j^2,\end{aligned}$$

whereas for $2 \leq k \leq k_0$, there holds

$$\begin{aligned}\left\langle \left(\sum_{j=1}^{k_0} u_j e_j \right)^2, e_k \right\rangle &= \sum_{i,j=1}^{k_0} u_i u_j \langle e_j e_i, e_k \rangle = 2u_1 \sum_{j=1}^{k_0} u_j \langle e_j e_1, e_k \rangle + \sum_{i,j=2}^{k_0} u_i u_j \langle e_j e_i, e_k \rangle \\ &= 2(2a)^{-1/2} u_1 u_k + \sum_{i,j=2}^{k_0} \eta^k_{i,j} u_i u_j,\end{aligned}$$

as only the term with $j = k$ in the first sum is non-zero. □

1.3 Slow and Galerkin manifolds

In analogy to standard procedure for fast-slow ODEs of singular perturbation type, the first step in our geometric analysis is to determine the critical manifold for (1.2). Considering the slow formulation of (1.2), obtained from the time rescaling $\tau = \varepsilon t$,

$$\varepsilon u_\tau = u_{xx} - v + u^2 + H^u(u, v, \varepsilon) \qquad \text{for } x \in (-a, a) \text{ and } \tau > 0, \tag{1.12a}$$

$$v_\tau = v_{xx} - 1 + H^v(v^2) \qquad \text{for } x \in (-a, a) \text{ and } \tau > 0, \tag{1.12b}$$

$$u_x(\tau, x) = 0 = v_x(\tau, x) \qquad \text{for } x = \mp a \text{ and } \tau > 0, \tag{1.12c}$$

and setting $\varepsilon = 0$ therein, we find that the critical manifold is given by the set

$$\left\{(u, v) : 0 = u_{xx} - v + u^2 + H^u(u, v, 0),\ u_x(\cdot, \mp a) = 0 = v_x(\cdot, \mp a)\right\}. \tag{1.13}$$

Restricting to spatially homogeneous solutions, we define the critical manifold S_0 as the set of functions

$$S_0 := \left\{(u, v) \in \mathbb{R}^2 : 0 = -v + u^2 + H^u(u, v, 0)\right\}, \tag{1.14}$$

abusing notation and identifying constant functions $u : [-a, a] \to \mathbb{R}$ with the value u they take.

Remark 1.4. If we do not restrict our attention to only constant functions, formally the critical manifold will be infinite-dimensional and given by the graph

$$\{(u, v) : v = u_{xx} + u^2 + H^u(u, v, 0), u_x(\cdot, \mp) = 0 = v_x(\cdot, \mp a)\}.$$

Studying this extended critical manifold would correspond, as discussed in the introduction, to designularising along a surface of the discretised system (1.8) which would be an interesting undertaking for future work.

Due to our assumptions on the form of H^u, near the origin $(u, v) = (0, 0)$ in (u, v)-space the set S_0 is given as a graph

$$S_0 = \left\{(u, v) \in \mathbb{R}^2 : v = u^2 + \mathcal{O}\left(u^3\right)\right\}. \tag{1.15}$$

Proceeding, again, as in a finite-dimensional setting, the second step in our analysis concerns the persistence of the manifold S_0 for ε positive and sufficiently small. However, in an infinite-dimensional setting, the concept of "fast" and "slow" variables can be delicate, as for any $\varepsilon > 0$, there exists $k > 0$ such that $\varepsilon \lambda_k = O(1)$. One way to address this complication is to split the slow variable v into fast and slow parts, which we make precise in the following proof of Proposition 1.5. In short, this splitting is facilitated by the parameter $\zeta \approx 1/k^2$, in the sense that the space Y_S^ζ in the proposition below is $\lfloor \zeta^{-1/2} \rfloor$-dimensional. We refer to [30] for further discussion and details.

Proposition 1.5. *Let $(u, v) \in S_0$ with $u < 0$. Consider any small $\zeta > 0$ and $u \le \omega_A < 0$, $\omega_f \in \mathbb{R}$, and $L_f > 0$ such that $\omega_A + L_f < \omega_f < 0$. Then, there exist spaces $Y_S^\zeta \oplus Y_F^\zeta = L^2(-a, a)$, with Y_S^ζ finite-dimensional, and a family of attracting slow manifolds around (u, v) that are given as graphs*

$$S_{\varepsilon,\zeta} := \left\{\left(h_X^{\varepsilon,\zeta}(v), h_{Y_F^\zeta}^{\varepsilon,\zeta}(v), v\right) : v \in Y_S^\zeta\right\} \tag{1.16}$$

for $0 < \varepsilon < C\frac{\omega_f}{\omega_A}\zeta$ and some fixed $C \in (0, 1)$, where $\left(h_X^{\varepsilon,\zeta}(v), h_{Y_F^\zeta}^{\varepsilon,\zeta}(v)\right) : Y_S^\zeta \to H^2(-a, a) \times \left(Y_F^\zeta \cap H^2(-a, a)\right)$.

Proof. We show that the assumptions of [39, Theorem 2.4] are satisfied. which will imply the existence of a family of slow manifolds stated in (1.16). Given a point $(u, v) = (c, c^2 + \mathcal{O}(c^3))$ on S_0, with $c < 0$ negative, we first translate that point to the origin in (1.2), which yields

$$u_t = u_{xx} - v + u^2 + 2cu + \widetilde{H}^u(u, v, \varepsilon) \qquad \text{for } x \in (-a, a) \text{ and } t > 0, \tag{1.17a}$$

$$v_t = \varepsilon\left(v_{xx} - 1 + H^v(v^2)\right) \qquad \text{for } x \in (-a, a) \text{ and } t > 0, \tag{1.17b}$$

$$u_x(t, x) = 0 = v_x(t, x) \qquad \text{for } x = \mp a \text{ and } t > 0. \tag{1.17c}$$

Here, $\widetilde{H}^u$ are new higher-order terms that are obtained from H^u post-translation. We choose

$$X = L^2(-a, a) \quad \text{and} \quad Y = L^2(-a, a) \tag{1.18}$$

as the basis spaces for u and v, respectively, and consider $X_\alpha = H^{2\alpha}(-a,a)$ and $Y_\alpha = H^{2\alpha}(-a,a)$ for $\alpha \in [0,1)$ (H^s here denotes the fractional Sobolev spaces; see [1]). The linear operators L_1 and L_2 are defined as $L_1 u = u_{xx} + 2cu$ and $L_2 = v_{xx}$, respectively, with $\mathscr{D}(L_1) = \mathscr{D}(L_2) = \{\phi \in H^2(-a,a) \, : \, \phi_x(-a) = 0 = \phi_x(a)\}$.

Since we are interested in a neighbourhood of the origin in (1.17) and by rescaling $v = \kappa_v \tilde{v}$, for any $\kappa_v > 0$, we consider the modified nonlinear terms

$$f(u,v) = -\kappa_v v + \chi(u)u^2 + \chi(u)\chi(v)\widehat{H}^u \quad \text{and} \tag{1.19a}$$

$$g(u,v) = -1/\kappa_v + \chi(u)\chi(v)\widehat{H}^v, \tag{1.19b}$$

where $\chi : H^2(-a,a) \to [0,1]$ is such that

$$\chi(u) = 1 \ \text{if} \ \|u\|_{H^2} \le \sigma^2, \quad \chi(u) = 0 \ \text{if} \ \|u\|_{H^2} \ge 2\sigma, \quad \text{and} \quad \|D\chi\|_{\mathscr{L}(H^2,\mathbb{R})} \le \sigma$$

for some $0 < \sigma < 1$, $\widehat{H}^u$ and $\widehat{H}^v$ denote the higher-order terms with rescaled v, where the tilde in v is omitted. Then, these modified nonlinearities

$$f : H^2(-a,a) \times L^2(-a,a) \to L^2(-a,a) \quad \text{and} \quad g : H^2(-a,a) \times H^2(-a,a) \to H^2(-a,a) \tag{1.20}$$

satisfy

$$\begin{aligned} \|Df(u,v)\|_{\mathscr{L}(H^2\times L^2, L^2)} &\le L_{f_1}, \\ \|Df(u,v)\|_{\mathscr{L}(H^2\times H^2, H^2)} &\le L_{f_2}, \quad \text{and} \\ \|Dg(u,v)\|_{\mathscr{L}(H^2\times H^2, H^2)} &\le L_g, \end{aligned} \tag{1.21}$$

where $\mathscr{L}(V,W)$ is the space of linear operators from V into W. Define $L_f := \min\{L_{f_1}, L_{f_2}\}$ and note that, by choosing $\sigma > 0$ small, the constants L_f and L_g can be made appropriately small.

Note also that, for any $\varepsilon > 0$, there exists $k > 0$ such that $\varepsilon\lambda_k = \mathscr{O}(1)$, where $\lambda_k = -\frac{k^2\pi^2}{4a^2}$, $k = 0,1,\dots$ are the eigenvalues of the operator L_2 with zero Neumann boundary conditions. Thus, to define fast and slow variables, we need to split the basic space $Y = L^2(-a,a)$ for v into $Y = Y_S^\zeta \oplus Y_F^\zeta$, where

$$Y_S^\zeta := \operatorname{span}\{e_k(x) : 0 \le k \le k_0\} \quad \text{and} \tag{1.22a}$$

$$Y_F^\zeta := \overline{\operatorname{span}\{e_k(x) : k > k_0\}}^{L^2}, \tag{1.22b}$$

with $\{e_k(x)\}_{k\in\mathbb{N}}$ being the eigenfunctions of the operator L_2 corresponding to the eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ and $k_0 \in \mathbb{N}$ satisfying

$$-\frac{(k_0+1)^2\pi^2}{4a^2} < \zeta^{-1}\omega_A \le -\frac{k_0^2\pi^2}{4a^2}, \tag{1.23}$$

for given $\zeta > 0$ and $\omega_A \in (2c, 0)$.

Then, for the semigroups generated by $-B_S$ and B_F, which are the realisations of the operator L_2 in $Y_S^\zeta \cap L^2(-a,a)$ and $Y_F^\zeta \cap L^2(-a,a)$, respectively, we have the following estimates:

$$\|e^{-tB_S} y_S\|_{H^2} \le e^{\frac{\pi^2 k_0^2}{4a^2} t} \|y_S\|_{H^2} \qquad \text{for } y_S \in Y_S^\zeta, \tag{1.24a}$$

$$\|e^{tB_F} y_F\|_{H^2} \le e^{-\frac{\pi^2(k_0+1)^2}{4a^2}t}\|y_F\|_{H^2} \qquad \text{for } y_F \in Y_F^\zeta \cap H^2(-a,a), \tag{1.24b}$$

see e.g. [26, p.20].

Now, using (1.22) and the estimates in (1.24) and following the proof of [39, Theorem 2.4] and [30], we obtain the stated results. The proof itself is a standard Lyapunov-Perror argument that requires careful estimates. The key step is the splitting that gives a spectral gap allowing the construction of $S_{\varepsilon,\zeta}$ as a graph over Y_S^ζ. □

Remark 1.6. Here, we have written Y_S^ζ instead of $Y_S^\zeta \cap H^2(-a,a)$, as Y_S^ζ is a finite-dimensional subspace of $H^2(-a,a)$. In general, Y_S^ζ is a subspace of L^2 and one has to be careful when taking the intersection with H^2.

Next, for given $\zeta > 0$, we also split the space $X = L^2(-a,a)$ into $X = X_S^\zeta \oplus X_F^\zeta$, where X_S^ζ and X_F^ζ are defined in the same manner as Y_S^ζ and Y_F^ζ, see (1.22).

Then, the truncation of the Galerkin system in (1.7) at k_0, which is related to ζ via (1.23), gives the projection of (1.17) onto $\left(X_S^\zeta, Y_S^\zeta\right)$. Thus, we obtain a family of so-called Galerkin manifolds

$$G_{\varepsilon,\zeta} := \left\{\left(h_G^{\varepsilon,\zeta}(v), v\right) : v \in Y_S^\zeta\right\} \tag{1.25}$$

for a function $h_G^{\varepsilon,\zeta} : Y_S^\zeta \to X_S^\zeta$.

Proposition 1.7. *For $0 < \varepsilon < C\frac{\omega_f}{\omega_A}\zeta$ and some fixed $C \in (0,1)$, where ζ, ω_A and ω_f are as in Proposition 1.5, the following estimate holds:*

$$\left\|h_X^{\varepsilon,\zeta}(v) - h_G^{\varepsilon,\zeta}(v)\right\|_{H^2} + \left\|h_{Y_F^\zeta}^{\varepsilon,\zeta}(v)\right\|_{H^2} \le \tilde{C}\left(\frac{4a^2}{\pi^2(2k_0+1)} + \zeta\right)\|v\|_{H^2}. \tag{1.26}$$

In particular, using the relation between ζ and k_0 in (1.23), we have

$$\left\|h_X^{\varepsilon,\zeta}(v) - h_G^{\varepsilon,\zeta}(v)\right\|_{H^2} + \left\|h_{Y_F^\zeta}^{\varepsilon,\zeta}(v)\right\|_{H^2} \le \tilde{C}\frac{1}{k_0}\|v\|_{H^2}. \tag{1.27}$$

Proof. See [39, Theorem 3.1]. □

Remark 1.8. Note that $k_0 \to \infty$ corresponds to $\zeta \to 0$ which, due to the relation $0 < \varepsilon < C\frac{\omega_f}{\omega_A}\zeta$, see Propositions 1.5 and 1.7, implies also $\varepsilon \to 0$ when $k_0 \to \infty$. Hence, the limit of the Galerkin manifolds $G_{\varepsilon,\zeta}$ as $k_0 \to \infty$ cannot, in general, be guaranteed for all $0 < \varepsilon < \varepsilon_0$ with an independent upper bound ε_0. Thus, we perform the following analysis for $0 < \varepsilon < \varepsilon_0$, with ε_0 sufficiently small, and k_0 arbitrarily large, but fixed.

1.4 Fast-slow analysis

Consider an arbitrary, fixed $k_0 \in \mathbb{N}$ in Proposition 1.2. A rescaling of the variables in (1.8) via $\hat{u}_k = a^{-1/2} u_k$ and $\hat{v}_k = a^{-1/2} v_k$ gives the fast-slow system

$$u_1' = -v_1 + 2^{-1/2} u_1^2 + 2^{-1/2} \sum_{j=2}^{k_0} u_j^2 + H_1^u, \tag{1.28a}$$

$$v_1' = -2^{1/2} \varepsilon, \tag{1.28b}$$

$$u_k' = \frac{1}{4} a^{-2} b_k u_k - v_k + 2^{1/2} u_1 u_k + \sum_{i,j=2}^{k_0} \eta_{i,j}^k u_i u_j + H_k^u, \tag{1.28c}$$

$$v_k' = \frac{1}{4} a^{-2} b_k \varepsilon v_k + \varepsilon H_k^v \tag{1.28d}$$

for $2 \le k \le k_0$. The rescaled system in (1.28) is equivalent to the original one in (1.8), in that orbits of the latter are mapped to those of the former by a smooth map. Thus, without loss of generality, in our analysis, we will henceforth focus on (1.28). In the slow time variable $\tau = \varepsilon t$, Equation (1.28) becomes

$$\varepsilon \dot{u}_1 = -v_1 + 2^{-1/2} u_1^2 + 2^{-1/2} \sum_{j=2}^{k_0} u_j^2 + H_1^u, \tag{1.29a}$$

$$\dot{v}_1 = -2^{1/2}, \tag{1.29b}$$

$$\varepsilon \dot{u}_k = \frac{1}{4} a^{-2} b_k u_k - v_k + 2^{1/2} u_1 u_k + \sum_{i,j=2}^{k_0} \eta_{i,j}^k u_i u_j + H_k^u, \tag{1.29c}$$

$$\dot{v}_k = \frac{1}{4} a^{-2} b_k v_k + H_k^v, \tag{1.29d}$$

with the overdot denoting differentiation with respect to τ.

Recalling the system of PDEs in (1.2), where the singularity is located at the origin of $L^2(-a, a)$, we will be considering initial data in a neighbourhood of that origin in the L^2-norm, with

$$\sum_{k=1}^{\infty} |u_k(0)|^2 \le \kappa \quad \text{and} \quad \sum_{k=1}^{\infty} |v_k(0)|^2 \le \kappa, \tag{1.30}$$

where $0 < \kappa < 1$. In addition, we impose the bounds

$$|u_k(0)| \le C_{k,u_0} \quad \text{and} \quad |v_k(0)| \le C_{k,v_0} \varepsilon^{4/3} \quad \text{for } k = 2, 3, \ldots, k_0, \tag{1.31}$$

where C_{k,u_0} and C_{k,v_0} are positive constants. The initial conditions for the first mode $\{u_1, v_1\}$ are taken as in the finite-dimensional (planar) case [34], and are specified in (1.38) below.

The assumption in (1.31) implies that the higher-order modes $u_k(0)$ and $v_k(0)$, corresponding to non-constant eigenfunctions, are sufficiently small. The requirement that $v_k(0)$ is of the order $\mathcal{O}(\varepsilon^{4/3})$ is essential for ensuring that $v_k(t)$ does not exhibit finite-time blowup before transiting through a neighbourhood of the singularity at the origin; see Section 1.4.3 for an example and Lemma 1.36 for details.

1.4.1 Critical manifold

Clearly, the system in (1.28) is a fast-slow system in the standard form of GSPT, with ε the (small) singular perturbation parameter and $\{u_k\}$ and $\{v_k\}$, $k = 1, 2, \dots, k_0$, the fast and slow variables, respectively. The critical manifold $\mathscr{C}$ for (1.28) is hence given as a graph over $(u_1, u_2, \dots, u_{k_0})$, with

$$v_1 = f_1(u_1, u_2, \dots, u_{k_0}) := 2^{-1/2} u_1^2 + 2^{-1/2} \sum_{j=2}^{k_0} u_j^2 \quad \text{and} \tag{1.32a}$$

$$v_k = f_k(u_1, u_2, \dots, u_{k_0}) := \frac{1}{4} a^{-2} b_k u_k + 2^{1/2} u_1 u_k + \sum_{i,j=2}^{k_0} \eta_{i,j}^k u_i u_j \text{ for } k = 2, \dots, k_0. \tag{1.32b}$$

Note that, in general, $\mathscr{C}$ is not normally hyperbolic: it contains attracting and saddle-type regions, as well as non-hyperbolic sets separating those regions; examples can be found in subsection 1.4.3. Of particular interest is the submanifold $\mathscr{C}_0 \subset \mathscr{C}$ of the critical manifold $\mathscr{C}$ which is defined as

$$\mathscr{C}_0 := \big\{\big(u_1, \dots, u_{k_0}, f_1(u_1, \dots, u_{k_0}), \dots, f_{k_0}(u_1, \dots, u_{k_0}\big) \in \mathscr{C} : \ u_1 < 0 \text{ and } u_k = 0 \text{ for } 2 \le k \le k_0\big\}. \tag{1.33}$$

In other words, $\mathscr{C}_0$ is obtained by setting $u_k = 0$ for $k = 2, \dots, k_0$ in (1.32), and can hence be written as the curve

$$\mathscr{C}_0 = \big\{(u_1, \dots, u_{k_0}, v_1, \dots, v_{k_0}) \in \mathbb{R}^{2k_0} : v_1 = 2^{-1/2} u_1^2 + \mathscr{O}\left(u_1^3\right), \text{ with } u_1 < 0 \text{ and } u_k = 0 = v_k \text{ for } 2 \le k \le k_0\big\} \tag{1.34}$$

that lies in the (u_1, v_1)-plane. The set $\mathscr{C}_0$ corresponds directly to the set of constant functions S_0, given by (1.14). We will denote the slow manifold that is obtained from $\mathscr{C}_0$ via GSPT by $\mathscr{C}_\varepsilon$ – or by $\mathscr{C}_{\varepsilon,k_0}$, to emphasise the dependence thereof on k_0.

Remark 1.9. Note that in Section 1.2, both Y_S^ζ and X_S^ζ are finite-dimensional, and that $G_{\varepsilon,\zeta}$ can hence be viewed as the Fenichel slow manifold perturbing off the normally hyperbolic subset $\mathscr{C}_0$ of the critical manifold of the fast-slow system in (1.28), for k_0 defined by ζ through (1.23).

Lemma 1.10. *The subset $\mathscr{C}_0$ of the critical manifold $\mathscr{C}$ is normally hyperbolic and attracting under the layer flow that is obtained for $\varepsilon = 0$ in* (1.28).

Proof. Linearising the fast flow of (1.28) around $\mathscr{C}_0$, we find the Jacobian matrix

$$\operatorname{diag}\left\{2^{1/2} u_1, 2^{1/2} u_1 + \frac{1}{4} a^{-2} b_2, \dots, 2^{1/2} u_1 + \frac{1}{4} a^{-2} b_{k_0}\right\}, \tag{1.35}$$

which implies that $\mathscr{C}_0$ is normally hyperbolic and attracting for $u_1 < 0$ bounded away from zero. (Recall that $b_k < 0$ for $k \in \mathbb{N}$.) □

Since the eigenvalues of a matrix depend continuously on its entries, and since the eigenvalues of $\mathscr{C}_0$ are all strictly negative, there exists a full neighbourhood around $\mathscr{C}_0$ in $\mathscr{C}$ which is normally hyperbolic and attracting under the layer flow of

(1.28). The flow in that neighbourhood is directed towards the origin where, as can be seen from the above linearisation, normal hyperbolicity is lost and which is hence a partially degenerate steady state of (1.28). The description of the dynamics near the origin therefore necessitates the application of geometric desingularisation.

1.4.2 Statement of main result

We are now ready to formulate our main result, which concerns the transition between two appropriately defined sections Δ^{in} and Δ^{out} for the flow generated by (1.28). These sections of the phase space are defined as follows: consider the set

$$\{(u_1, v_1) \,:\, u_1 \in J \text{ and } v_1 = \rho^2\} \subset \mathbb{R}^{k_0} \times \mathbb{R}^{k_0}, \tag{1.36}$$

for small $\rho > 0$ and a suitable interval J, and let Δ^{in} be a neighbourhood of this set in $\mathbb{R}^{k_0} \times \mathbb{R}^{k_0}$. Similarly, define Δ^{out} as a neighbourhood of the set

$$\{(u_1, v_1) \,:\, u_1 = \rho \text{ and } v_1 \in \mathbb{R}\} \subset \mathbb{R}^{k_0} \times \mathbb{R}^{k_0} \tag{1.37}$$

that is contained in the (u_1, v_1)-plane. More explicitly, let

$$\begin{aligned}\Delta^{\text{in}} = \Big\{&(u_1, \dots, u_{k_0}, v_1, \dots, v_{k_0}) \in \mathbb{R}^{2k_0} : u_1 \in \left(-2^{1/4}\rho - C^{\text{in}}_{u_1}, -2^{1/4}\rho + C^{\text{in}}_{u_1}\right), \\ &v_1 = \rho^2, |u_k| \le C^{\text{in}}_{u_k}, \text{ and } |v_k| \le C^{\text{in}}_{v_k} \text{ for } 2 \le k \le k_0\Big\}\end{aligned} \tag{1.38}$$

and

$$\begin{aligned}\Delta^{\text{out}} = \Big\{&(u_1, \dots, u_{k_0}, v_1, \dots, v_{k_0}) \in \mathbb{R}^{2k_0} : u_1 = \rho, \\ &v_1 \in \mathbb{R}, |u_k| \le C^{\text{out}}_{u_k}, \text{ and } |v_k| \le C^{\text{out}}_{v_k} \text{ for } 2 \le k \le k_0\Big\},\end{aligned} \tag{1.39}$$

where $C^{\text{in}}_{u_1}$, $C^{\text{in}}_{u_k}$, $C^{\text{in}}_{v_k}$, $C^{\text{out}}_{u_k}$, and $C^{\text{out}}_{v_k}$ $(2 \le k \le k_0)$ are appropriately chosen, small constants. Given these definitions, we have the following result on the transition map between the sections Δ^{in} and Δ^{out} that is induced by the flow of (1.28). The main result of the chapter, proven in section 1.5, is the following.

Theorem 1.11. *Fix $k_0 \in \mathbb{N}$, and consider the subset $R^{\text{in}}_1 \subset \Delta^{\text{in}}$ defined by*

$$\begin{aligned}R^{\text{in}} = R^{\text{in}}(\varepsilon) := \Big\{&(u_1, \dots, u_{k_0}, v_1, \dots, v_{k_0}) \in \mathbb{R}^{2k_0} : u_1 \in \left(-2^{1/4}\rho - C^{\text{in}}_{u_1}, -2^{1/4}\rho + C^{\text{in}}_{u_1}\right), \\ &v_1 = \rho^2, |u_k| \le C^{\text{in}}_{u_k}, \text{ and } |v_k| \le C^{\text{in}}_{v_k}\varepsilon^{4/3} \text{ for } 2 \le k \le k_0\Big\}.\end{aligned} \tag{1.40}$$

Then, there exists ε_0 (that depends on k_0) such that for $0 < \varepsilon < \varepsilon_0$, the system in (1.28) *admits a well-defined transition map*

$$\Pi : R^{\text{in}} \to \Delta^{\text{out}}.$$

Let $(u^{\text{in}}_1, v^{\text{in}}_1, u^{\text{in}}_k, v^{\text{in}}_k) \in R^{\text{in}}$ and

$$(u^{\text{out}}_1, v^{\text{out}}_1, u^{\text{out}}_k, v^{\text{out}}_k) := \Pi(u^{\text{in}}_1, v^{\text{in}}_1, u^{\text{in}}_k, v^{\text{in}}_k);$$

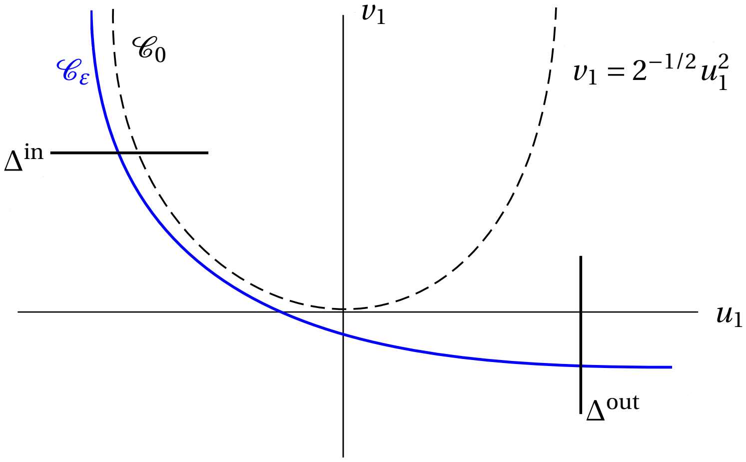


Figure 1.1: Illustration of the main result, Theorem 1.11, in its projection onto the (u_1, v_1)-plane. The sections Δ^{in} and Δ^{out} are, in fact, full neighbourhoods around the shown line intervals in u_1 and v_1. Given $k_0 \in \mathbb{N}$ fixed, trajectories of (1.28) that are initiated in Δ^{in} will intersect Δ^{out} transversely for ε sufficiently small.

then,

$$\begin{aligned} |v_1^{\text{out}}| &= \mathcal{O}\left(\varepsilon^{2/3}\right), & u_1^{\text{out}} &= \rho, \\ |u_k^{\text{out}}| &\leq C|u_k^{\text{in}}|, \quad \textit{and} & |v_k^{\text{out}}| &\leq C|v_k^{\text{in}}| \end{aligned} \tag{1.41}$$

for $2 \leq k \leq k_0$ and positive generic constants C and c which may differ between estimates. In particular, the slow manifolds $\mathscr{C}_\varepsilon$ cross the section Δ^{out} transversely. In addition, Π restricted to $I := \{(u_1, u_k, v_1, v_k) \in R^{\text{in}}$ for fixed $u_k, v_k, 2 \leq k \leq k_0\}$, is a contraction with rate $e^{-c/\varepsilon}$ for a constant $c < 0$.

Remark 1.12. In (1.28), the equations for (u_1, v_1) reduce to those for the classical singularly perturbed planar fold [34] if we set $u_k = 0$ for $2 \leq k \leq k_0$. Here, we perform a similar analysis as there while controlling the higher-order modes $\{u_k, v_k\}$, $2 \leq k \leq k_0$. Note that we are restricting to initial data for the system of PDEs in (1.2) that are close to constant functions, which translates to small initial data $\{u_k(0), v_k(0)\}$ for the system of ODEs in (1.28).

As mentioned already, the dependence on ε in the initial values for v_k, $2 \leq k \leq k_0$, is essential to ensure that trajectories of the Galerkin system in (1.28) do not exhibit finite time blowup before reaching Δ^{out}; see the following example and the corresponding estimates in section 1.5.

1.4.3 Illustrative example: $k_0 = 2$

In order to develop intuition for the singular geometry and resulting dynamics of (1.28), it is instructive to first examine the simple case where $k_0 = 2$. For simplicity, let $a = \frac{1}{2}$, and assume that the higher-order terms H_i^u and H_i^v for $i = 1, 2$ are identically zero. In that case, the system in (1.28) reads

$$u_1' = -v_1 + 2^{-1/2}u_1^2 + 2^{-1/2}u_2^2, \tag{1.42a}$$
$$v_1' = -2^{-1/2}\varepsilon, \tag{1.42b}$$
$$u_2' = -\pi^2 u_2 - v_2 + 2^{1/2}u_1u_2, \tag{1.42c}$$
$$v_2' = -\pi^2\varepsilon v_2, \tag{1.42d}$$

where the critical manifold $\mathscr{C}$ is given by the graph

$$v_1 = f_1(u_1,u_2) := 2^{-1/2}u_1^2 + 2^{-1/2}u_2^2 \quad \text{and} \tag{1.43a}$$
$$v_2 = f_2(u_1,u_2) := \pi^2 u_2 + 2^{1/2}u_1u_2. \tag{1.43b}$$

Linearisation of the layer problem induced by (1.43) for $\varepsilon = 0$ about $\mathscr{C}$ reveals that one eigenvalue is always negative for any choice of (u_1,u_2), whereas the sign of the other eigenvalue depends on (u_1,u_2), as shown in Figure 1.2. The set $\mathscr{C}_0$ is denoted in red there. To the left of the curve $u_1 = g(u_2) := \frac{1}{2}\left(\pi^2 - \sqrt{\pi^2 + 4u_2^2}\right)$ (illustrated in blue), the second eigenvalue is negative, whereas it is positive to the right of the curve. Normal hyperbolicity is lost on the curve itself.

Remark 1.13. Due to the presence of the line of non-hyperbolic points $u_1 = g(u_2)$, of which the origin $(0,0)$ is part of, we are not in the presence of a normal form fold point in the sense of [34], as further singularities surround the origin. This is the reason that the estimates given given in the next sections are required in order to control their influence.

Remark 1.14. Similarly, one can visualise the stability properties of the critical manifold $\mathscr{C}$ in the case where $k_0 = 3$, which will be given as a graph over (u_1,u_2,u_3); see Figure 1.3. Specifically, the manifold $\mathscr{C}$ is then attracting inside the funnel-like region of (u_1,u_2,u_3)-space shown in the figure and of saddle type outside that region. In analogy to the case of $k_0 = 2$, normal hyperbolicity is lost on the surface separating those two regions which is now given by an implicit polynomial expression that can be obtained by application of the Routh-Hurwitz stability criterion. The set $\mathscr{C}_0$ is again drawn in red.

For this particular case, it is possible to find explicit formulae for the initial conditions which will allow us to reach the section Δ^{in} under the flow of (1.42). Firstly, (u_1,u_2) must be in the region of the (u_1,u_2)-plane that corresponds to the normally hyperbolic attracting part of the critical manifold $\mathscr{C}$; recall Figure 1.2. Secondly, by GSPT, we have to be sufficiently close to the corresponding slow manifold $\mathscr{C}_\varepsilon$ for ε sufficiently small, which amounts to a condition of the form

$$\max\{|v_1 - f_1(u_1,u_2)|, |v_2 - f_2(u_1,u_2)|\} < C, \tag{1.44}$$

where $C > 0$ is some suitably chosen constant.

Thirdly, we also need to impose corresponding restrictions on (v_1,v_2) to ensure that we will not reach an unstable part of the critical manifold C under the slow flow induced by (1.42). To that end, we first need to invert the line $u_1 = g(u_2)$ which separates the attracting and saddle-like parts of C in the (u_1,u_2)-plane by substituting

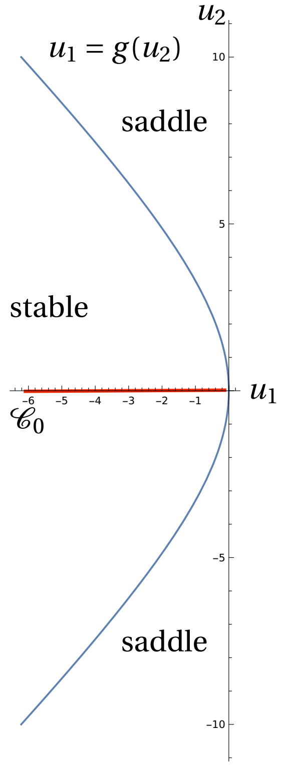


Figure 1.2: Stability properties of the critical manifold $\mathscr{C}$ which, for $k_0 = 2$, can be written as a graph over (u_1, u_2). A loss of normal hyperbolicity occurs along the curve $u_1 = g(u_2)$ (in blue) where one of the two eigenvalues of the linearisation about $\mathscr{C}$ changes sign. The manifold $\mathscr{C}_0$ is shown in red.

into (1.43) and solving for v_1 and v_2. The result is now a curve in the (v_1, v_2)-plane of the form $v_1 = h(v_2)$, where the function h is a quadratic polynomial in v_2, as shown in blue in Figure 1.4.

A further, fourth, restriction is given by solving explicitly the (v_1, v_2)-subsystem in (1.42) – rewritten in terms of the slow time – and by then determining a relation between v_1 and v_2 so that the flow reaches the section Δ^{in}:

$$|v_2| \leq \xi(v_1) := C_{v_2}^{\text{in}} e^{\frac{\pi^2}{\sqrt{2}}(v_1 - \rho^2)}; \tag{1.45}$$

see Figure 1.4 (in purple) for an illustration.

Initial values for (1.42) satisfying these four conditions will flow into the section Δ^{in}; however, as the flow of (1.42) approaches the singularity at the origin, u_2 may blow up before v_1 becomes negative. The flow will hence have reached an unstable part of the critical manifold $\mathscr{C}$. Next, we provide an explicit explanation for this blowup in finite time.

Finite time blowup of solutions

Solutions of the Galerkin system in (1.8) and, correspondingly, of the transformed system in (1.28), may blow up in finite time. Indeed, we prove in the following that a finite-time blowup can already occur for $k_0 = 2$. To demonstrate this, we work with

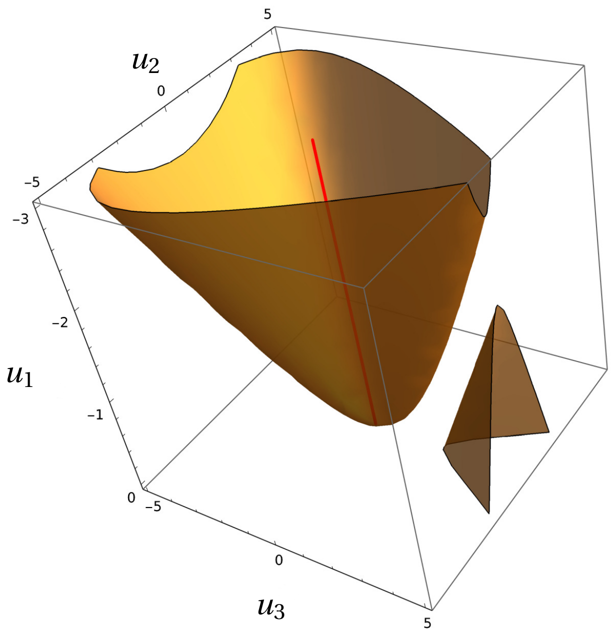


Figure 1.3: The fold surface for $k_0 = 3$. The critical manifold $\mathscr{C}$ is a graph over (u_1, u_2, u_3) and is stable inside the funnel-like region, the boundary of which is a surface that is implicitly defined by a polynomial equation in u_1, u_2, and u_3. One of the three eigenvalues of the linearisation about $\mathscr{C}$ changes sign across the surface.

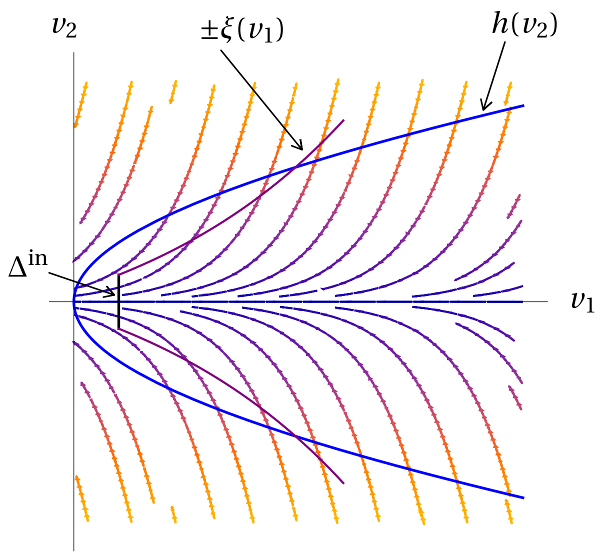


Figure 1.4: The reduced flow of (1.42). The region inside the curve $h(v_2)$ (in blue) corresponds to the stable part of the critical manifold $\mathscr{C}$; across that curve, one of the eigenvalues of the linearisation about $\mathscr{C}$ changes sign. Also illustrated are Δ^{in} (in black) and $\pm s(v_2)$ (in purple); recall (1.45). The set of initial conditions in the (v_1, v_2)-plane that reach Δ^{in} is found in the intersection of the regions to the right of $h(v_2)$ and $\pm s(v_2)$.

system (we could have worked with (1.42) directly but that would have been slightly more tedious)

$$\begin{aligned} u_1' &= -v_1 + u_1^2 + u_2^2, & u_1(0) &= u_1^0, \\ v_1' &= -\varepsilon, & v_1(0) &= v_1^0, \\ u_2' &= -v_2 + u_2(2u_1 - \pi^2), & u_2(0) &= u_2^0, \\ v_2' &= -\varepsilon\pi^2 v_2, & v_2(0) &= v_2^0. \end{aligned} \tag{1.46}$$

System (1.42) can be recovered by rescaling u_1 and u_2 by a factor of $2^{-1/2}$. It is assumed that $v_1^0 > 0$. We will show that, for $v_2^0 \neq 0$ and $\varepsilon > 0$ sufficiently small, a finite-time blowup will occur in (1.46) before v_1 changes sign.

Remark 1.15. The intuition for this finite-time blowup is that unless all four conditions described just above are satisfied, trajectories will cross the non-hyperbolic submanifold away from the neighbourhood of the origin we are analysing, leading to a finite-time blowup while v_1 is still negative. Of course, the solutions will still blowup in finite time even when the aforementioned conditions are satisfied but this will happen after v_1 becomes positive.

For the sake of simplicity and without loss of generality, we may assume that $u_2^0 < 0$ and $v_2^0 > 0$; see also Remark 1.22.

Proposition 1.16. *Let $u_1^0 \in \mathbb{R}$, $u_2^0 < 0$, and $v_1^0, v_2^0 > 0$. Then, there exists $\varepsilon > 0$ such that the solution of (1.46) blows up before $t_0 = \frac{v_1^0}{\varepsilon}$, i.e., before v_1 changes sign.*

Proof. As observed in Proposition 1.19 and Remark 1.22, without loss of generality, we may assume that

$$-\pi/2 < u_1^0 \leq \pi/4 \quad \text{and} \quad v_1^0 < \min\left\{\frac{\pi^2}{16}, \left(\frac{e^{-\pi^4/32} v_2^0}{2(\pi+\pi^2)}\right)^2\right\}.$$

By Proposition 1.20 and Proposition 1.21, the former also implies that $-\pi/2 < u_1(t) \leq \pi/4$ for all $t \geq 0$ unless there is a blowup in a finite time independent of $\varepsilon > 0$. We consider the time interval $[0, \frac{v_1^0}{2\varepsilon}]$ in which v_1 remains positive. Moreover, we have $v_2(t) \in \left[\exp\left(-\frac{\pi^2 v_1^0}{2}\right) v_2^0, v_2^0\right]$ for all $[0, \frac{v_1^0}{2\varepsilon}]$. Since $u_2 \leq 0$ by Remark 1.22 and since $-\pi/2 < u_1(t) \leq \pi/4$ for all $t \geq 0$, it follows that

$$-2v_2^0 - (\pi^2 - \tfrac{\pi}{2})u_2 < -v_2 + u_2(2u_1 - \pi^2) = \partial_t u_2 < -\exp\left(-\frac{\pi^2 v_1^0}{2}\right) v_2^0 - (\pi + \pi^2)u_2$$

in $\left[0, \frac{v_1^0}{2\varepsilon}\right]$. Let now w_u and w_o be the solutions of

$$\begin{aligned} w_u' &= -2v_2^0 - (\pi^2 - \tfrac{\pi}{2})w_u, \\ w_o' &= -\exp\left(-\frac{\pi^2 v_1^0}{2}\right) v_2^0 - (\pi + \pi^2)w_o, \end{aligned}$$

with $w_u(0) = u_2 = w_o(0)$, in $\left[0, \frac{v_1^0}{2\varepsilon}\right]$. It then follows from Lemma 1.18 that $w_u \le u_2 \le w_o$. Thus, in the interval $\left[\frac{v_1^0}{4\varepsilon}, \frac{v_1^0}{2\varepsilon}\right]$, we have

$$\begin{aligned} u_2(t) \le w_o(t) &= \exp\left(-(\pi+\pi^2)t\right)\left[u_2^0 + \frac{1}{(\pi+\pi^2)}\exp\left(-\frac{\pi^2 v_1^0}{2}\right)v_2^0\right] - \frac{1}{(\pi+\pi^2)}\exp\left(-\frac{\pi^2 v_1^0}{2}\right)v_2^0 \\ &\le -\frac{1}{2(\pi+\pi^2)}\exp\left(-\frac{\pi^2 v_1^0}{2}\right)v_2^0, \end{aligned}$$

provided $\varepsilon > 0$ is sufficiently small. Correspondingly, in $\left[\frac{v_1^0}{4\varepsilon}, \frac{v_1^0}{2\varepsilon}\right]$, we obtain that

$$\begin{aligned} u_1' = -v_1 + u_1^2 + u_2^2 \ge -v_1^0 + \frac{e^{-\pi^2 v_1^0}}{4(\pi+\pi^2)^2}\left(v_2^0\right)^2 + u_1^2 \ge -v_1^0 + \frac{e^{-\pi^4/16}}{4(\pi+\pi^2)^2}\left(v_2^0\right)^2 + u_1^2 \\ > c + u_1^2 \end{aligned} \tag{1.47}$$

for some $c > 0$ due to $v_1^0 < \min\left\{\frac{\pi^2}{16}, \left(\frac{e^{-\pi^4/32}v_2^0}{2(\pi+\pi^2)}\right)^2\right\}$. The equation

$$w' = \mu + w^2,$$

with $\mu > 0$ constant, experiences blowup for all initial conditions at a time t_0 that may depend on the initial condition, but that is independent of ε. If $\varepsilon > 0$ is small enough, then the blowup occurs in $\left[0, \frac{v_1^0}{4\varepsilon}\right]$. Thus, Lemma 1.18 implies that u_1 blows up before time $\frac{v_1^0}{2\varepsilon}$; in particular, it blows up before v_1 changes sign. □

Remark 1.17. It is possible to give an explicit estimate for the smallness of ε in terms of the initial conditions in system (1.46); see (1.49).

The following lemma was used in the proof of Proposition 1.16.

Lemma 1.18. *Let $f, g\colon [0,\infty)\times\mathbb{R} \to \mathbb{R}$ be such that $f(t,x) > g(t,x)$ for all $(t,x) \in [0,\infty)\times\mathbb{R}$, and suppose that f and g are locally Lipschitz continuous. Furthermore, let $x_0 \in \mathbb{R}$, and let y_f and y_g be the solutions of*

$$y_f'(t) = f(t, y_f(t)) \quad \textit{and} \quad y_g'(t) = g(t, y_g(t)), \quad \textit{with } y_f(0) = y_g(0).$$

Then, $y_f(t) \ge y_g(t)$ for all t in the intersection of the maximal existence intervals of y_f and y_g.

Proof. Assume that $y_f(t_0) = y_g(t_0)$ for some $t_0 \ge 0$. We will show that there exists $h > 0$ such that that $y_f(t) > y_g(t)$ for all $t \in [t_0, t_0+h]$ which proves the lemma.

Let $x_0 := y_f(t_0) = y_g(t_0)$. Since $f(t_0,x_0) > g(t_0,x_0)$, by the continuity of f, g, there exist $\zeta, \xi > 0$ such that $f(t_1,x_1) > g(t_2,x_2)$ for any $t_1, t_2 \in [t_0, t_0+\zeta]$ and $x_1, x_2 \in [x_0-\xi, x_0+\xi]$. Because the solutions y_f, y_g are continuous, there exists some $h > 0$ such that

$$|y_f(t) - y_f(t_0)| < \xi, \quad |y_g(t) - y_g(t_0)| < \xi, \quad \text{for all } t \in [t_0, t_0+h].$$

In addition, we can ensure that $h < \zeta$ by taking smaller h if required. Integrating then gives

$$y_f(t) - y_g(t) = \int_{t_0}^{t} \big(f(s, y_f(s)) - g(s, y_g(s)\big)\, ds > 0$$

for any $t \in [t_0, t_0 + h]$ and the proof is complete. □

Proposition 1.19. *If the solution of* (1.46) *exists for a sufficiently long time, then we may choose* $\eta > 0$ *independent of* ε *– but dependent on* v_1^0 *and* v_2^0 *– such that*

$$0 < v_1\big(\tfrac{v_1^0-\eta}{\varepsilon}\big) < \min\left\{\tfrac{\pi^2}{16}, \tfrac{e^{-\pi^4/16}}{4(\pi+\pi^2)^2}\left[v_2\big(\tfrac{v_1^0-\eta}{\varepsilon}\big)\right]^2\right\}.$$

Proof. Solving explicitly, we can write $v_1(t) = v_1^0 - \varepsilon t$ and $v_2(t) = \exp\big(-\varepsilon\pi^2 t\big)v_2^0$. Hence,

$$v_1\big(\tfrac{v_1^0-\eta}{\varepsilon}\big) = \eta > 0.$$

On the other hand, for $\eta > 0$ sufficiently small, we have

$$v_1\big(\tfrac{v_1^0-\eta}{\varepsilon}\big) = \eta < \tfrac{e^{-\pi^4/16}}{4(\pi+\pi^2)^2}\exp\big(-2\pi^2(v_1^0-\eta)\big)(v_2^0)^2 = \tfrac{e^{-\pi^4/16}}{4(\pi+\pi^2)^2}\left[v_2\big(\tfrac{v_1^0-\eta}{\varepsilon}\big)\right]^2.$$

Obviously, if $\eta > 0$ is small enough, it also holds that

$$v_1\big(\tfrac{v_1^0-\eta}{\varepsilon}\big) = \eta < \tfrac{\pi^2}{16},$$

which shows the assertion. □

Given Proposition 1.19, blowup in (1.46) can still occur in a time interval of length η/ε. Since η can be chosen independent of ε, that interval can be made arbitrarily large for ε sufficiently small. In particular, if we can now show that solutions of (1.46) blow up after a time which is independent of ε, then blowup will occur before v_1 changes sign if $\varepsilon > 0$ is small enough. By Proposition 1.19, we may assume that $v_1^0 < \min\left\{\tfrac{\pi^2}{16}, \left[\tfrac{e^{-\pi^4/32}v_2^0}{2(\pi+\pi^2)}\right]^2\right\}$.

Proposition 1.20. *If the solutions of* (1.46) *exist for a sufficiently long time, then there exists a time* $t_0 \geq 0$*, which is independent of* ε*, such that* $u_1(t_0) > -\pi/2$.

Proof. Since we may assume $v_1^0 < \frac{\pi^2}{16}$, it holds that

$$u_1' = -v_1 + u_1^2 + u_2^2 > -\tfrac{\pi^2}{16} + u_1^2.$$

As long as $u_1^2 \leq -\pi/2$, we also have $-\frac{\pi^2}{16} + u_1^2 > \frac{3\pi^2}{16}$ and, hence, $u_1' > \frac{3\pi^2}{16}$, which proves the assertion. □

Proposition 1.21. *If* $u_1^0 > \pi/4$*, then solutions of* (1.46) *blow up after a finite time which is independent of* ε.

Proof. Since we may assume $v_1^0 < \frac{\pi^2}{16}$, it holds that

$$u_1' = -v_1 + u_1^2 + u_2^2 > -\tfrac{\pi^2}{16} + u_1^2.$$

If $u_1^0 > \pi/4$, then the right-hand side in the above expression is positive. It follows from (1.18) that a blowup occurs after a finite time which is independent of ε, as that is the case for the solution of

$$w' = -\tfrac{\pi^2}{16} + w^2, \quad w(0) = u_1^0 > \pi/4.$$

□

Remark 1.22. (a) Proposition 1.20 and Proposition 1.21 show that we may, without loss of generality, assume $-\pi/2 < u_1^0 \leq \pi/4$.

(b) One can also show the following: if solutions to (1.46) exist long enough and if $\varepsilon > 0$ is sufficiently small, then there exists a time $t_0 \geq 0$, independent of ε, such that $u_2(t) \leq 0$ for all $t \geq t_0$. We take $u_2^0 < 0$ and observe that this implies $u_2(t) \leq 0$ for all $t \geq 0$. Note however that one has to swap signs here if $v_2^0 < 0$.

These observations together allow us to prove Proposition 1.16. We now want to derive an estimate on how small ε has to be such that we observe blowup before v_1 changes sign. In a first step, we give an explicit expression for η – in dependence of v_1^0 and v_2^0, but not of ε – that satisfies the estimate in Proposition 1.19. For the sake of simplicity, we will assume that $v_1^0 \in (0, \pi^2/16)$.

Lemma 1.23. *If η is chosen as*

$$\eta = \frac{(v_2^0)^2}{4(\pi+\pi^2)^2} \exp\left(-\frac{\pi^4}{16} - 2\pi^2 v_1^0\right),$$

then the estimate in Proposition 1.19 is satisfied.

Proof. The estimate in Proposition 1.19 holds true if and only if

$$\begin{aligned} \eta = v_1^0 - \varepsilon\frac{v_1^0-\eta}{\varepsilon} = v_1\left(\frac{v_1^0-\eta}{\varepsilon}\right) &< \tfrac{e^{-\pi^4/16}}{4(\pi+\pi^2)^2}\left[v_2\left(\tfrac{v_1^0-\eta}{\varepsilon}\right)\right]^2 \\ = \tfrac{e^{-\pi^4/16}}{4(\pi+\pi^2)^2}(v_2^0)^2 \exp\left(-2\varepsilon\pi^2\frac{v_1^0-\eta}{\varepsilon}\right) &= \tfrac{e^{-\pi^4/16}}{4(\pi+\pi^2)^2}(v_2^0)^2 \exp\left[-2\pi^2(v_1^0-\eta)\right]. \end{aligned}$$

Multiplication by $e^{-2\pi^2\eta}$ yields

$$\eta \exp(-2\pi^2\eta) < \tfrac{e^{-\pi^4/16}}{4(\pi+\pi^2)^2}(v_2^0)^2 \exp(-2\pi^2 v_1^0),$$

which is satisfied if

$$\eta = \frac{(v_2^0)^2}{4(\pi+\pi^2)^2} \exp\left(-\tfrac{\pi^4}{16} - 2\pi^2 v_1^0\right),$$

as stated in the assertion. □

Remark 1.24. Note that, with the above choice of η, the estimate of Proposition 1.19 is satisfied at time

$$t_0 = \varepsilon^{-1}\left[v_1^0 - \frac{e^{-\pi^4/16}}{4(\pi+\pi^2)^2}\exp(-2\pi^2 v_1^0)(v_2^0)^2\right].$$

Following Proposition 1.19 and Lemma 1.23, we may thus replace v_1^0 by $v_1(\frac{v_1^0-\eta}{\varepsilon})=\eta$ and v_2^0 by $v_2(\frac{v_1^0-\eta}{\varepsilon})=\sqrt{\eta}e^{\pi^2\eta}\frac{2(\pi+\pi^2)}{e^{-\pi^4/32}}$ in Proposition 1.16. The fact that time t_0 has already passed is not significant to the arguments that follow.

Remark 1.25. The main argument in the proof of Proposition 1.16 was that solutions of

$$\dot{x}=\mu+x^2, \quad \text{with } x(0)=\xi, \tag{1.48}$$

blow up in finite time if $\mu>0$. The explicit solution of (1.48) is given by

$$x(t)=\sqrt{\mu}\tan\left(\arctan\left(\frac{\xi}{\sqrt{\mu}}\right)+\sqrt{\mu}t\right),$$

which hence exists until time

$$t=\frac{\pi/2-\arctan\left(\frac{\xi}{\sqrt{\mu}}\right)}{\sqrt{\mu}}.$$

In particular, blowup occurs before time $t=\pi/\sqrt{\mu}$. To determine how to choose μ in Proposition 1.16, we recall Equation (1.47), which allows for

$$\mu=-v_1^0+\frac{e^{-\pi^4/16}}{4(\pi+\pi^2)^2}(v_2^0)^2=(e^{2\pi^2\eta}-1)\eta;$$

here, we have used Lemma 1.23.

Proposition 1.26. *If* $\varepsilon<\frac{\eta^2}{2\sqrt{2}}$, *then the solution of* (1.46) *blows up before* v_1 *changes sign.*

Proof. In the proof of Proposition 1.16, blowup is generated in the time interval $[0,\frac{v_1^0}{4\varepsilon}]=[0,\frac{\eta}{4\varepsilon}]$. In combination with Remark 1.25, it follows that it suffices to take ε small enough such that $\varepsilon<\frac{\eta\sqrt{\mu}}{4\pi}$. To prove the assertion, we rewrite the right hand side of that inequality as

$$\frac{\eta\sqrt{\mu}}{4\pi}=\frac{\eta\sqrt{\eta(e^{2\pi^2\eta}-1)}}{4\pi},$$

which is, in fact, sharper than the right-hand side in the assertion; for conciseness, we observe that $e^{2\pi^2\eta}-1>2\pi^2\eta$, so that

$$\frac{\eta\sqrt{\mu}}{4\pi}>\frac{\eta^2}{2\sqrt{2}},$$

whence the assertion follows. □

Remark 1.27. (a) One can substitute the expression for η from Lemma 1.23 into the statement of Proposition 1.26 again in order to obtain an estimate in terms of the initial data v_1^0 and v_2^0, which yields

$$\varepsilon < \frac{e^{-\pi^4/8}}{32\sqrt{2}(\pi+\pi^2)^4}\exp\bigl(-4\pi^2 v_1^0\bigr)(v_2^0)^4. \tag{1.49}$$

While the resulting values of ε are tiny for most initial conditions v_1^0 and v_2^0, we note that for any choice of initial condition, there is an $\varepsilon_0(v_1^0, v_2^0)$ such that for any $\varepsilon \in [0, \varepsilon_0(v_1^0, v_2^0)]$, the solutions of (1.46) blow up before v_1 changes sign.

(b) Note that the estimate on ε in Proposition 1.26 is not sharp; specifically, our reliance on sandwich arguments and simplifications in the proof, as well as our choice of η, are not optimal. Given a particular systems, it may be numerically advantageous to refine these estimates accordingly.

1.5 Geometric desingularisation

Next, in order to prove our main result Theorem 1.11, we will describe the dynamics of the system of equations in (1.28) near the origin, which is a partially degenerate steady state. To that end, we will apply the method of geometric desingularisation by considering ε as a variable in (1.28) which is included in the quasi-homogeneous spherical coordinate transformation

$$u_k = \bar{r}^{\alpha_k}\bar{u}_k, \quad v_k = \bar{r}^{\beta_k}\bar{v}_k, \quad \text{and} \quad \varepsilon = \bar{r}^{\zeta}\bar{\varepsilon}; \tag{1.50}$$

here, $k = 1, 2, \dots, k_0$ and $(\bar{u}_1, \bar{v}_1, \dots, \bar{u}_{k_0}, \bar{v}_{k_0}, \bar{\varepsilon}) \in \mathbb{S}^{2k_0}$, with $\mathbb{S}^{2k_0}$ denoting the $2k_0$-sphere in $\mathbb{R}^{2k_0+1}$ and $r \in [0, r_0]$, for $r_0 > 0$ sufficiently small. The weights α_k, β_k, and ζ in (1.50) will be determined by a rescaling argument below.

In analogy to the desingularisation of the well-known planar fold via blow-up that is performed in [34], we shall work with three coordinate charts K_1, K_2, and K_3, which are formally obtained by setting $\bar{v}_1 = 1$, $\bar{\varepsilon} = 1$, and $\bar{u}_1 = 1$, respectively, in (1.50). As is convention, we will denote the variables corresponding to u_k, v_k, and ε in chart K_i $(i = 1, 2, 3)$ by $u_{k,i}$, $v_{k,i}$, and ε_i, respectively.

In a nutshell, our strategy will be to retrace the analysis in [34] in each of these charts; crucially, we will need to control the higher-order modes in (1.28), i.e., the variables u_k and v_k for $k = 2, \dots, k_0$, in the process. To be precise, we will verify that these additional variables will either remain uniformly bounded (in ε and k) or decay in the transition through the coordinate charts K_1, K_2, and K_3.

Before proceeding, let us briefly discuss one of the main result on the fold singularity in [34], as Theorem 1.11 can be viewed as a generalisation. The planar fold is the ODE system

$$u' = -v + u^2 + f(u, v, \varepsilon), \tag{1.51a}$$
$$v' = -\varepsilon g(u, v, \varepsilon), \tag{1.51b}$$

where $(u, v) \in \mathbb{R}^2$ and $f(u, v, \varepsilon) = \mathcal{O}(\varepsilon, uv, v^2, u^3)$, $g(u, v, \varepsilon) = -1 + \mathcal{O}(u, v, \varepsilon)$. The critical manifold is given, to the leading order, by the curve $\mathscr{C} := \{(u, v) \in \mathbb{R}^2 : v = u^2\}$. Letting $\mathscr{C}_0 = \{(u, v) \in \mathscr{C} : u < 0\}$ and since $\partial_u \bigl(-v + u^2 + f(u, v, 0)\bigr)|_{(u_0, v_0) \in \mathscr{C}_0} < 0$ due to our assumptions, $\mathscr{C}_0$ is normally hyperbolic attracting. It it shown in [34] that the perturbed slow manifold $\mathscr{C}_\varepsilon$ is extended by the flow of (1.51) as a curve parallel, to the leading order, to the u-axis. The picture is exactly as in Figure 1.1 if we set $k_0 = 1$ (up to rescaling by a factor of $2^{-1/2}$). As such, our main result can be viewed as a generalisation in the presence of k_0 higher order modes.

A significant challenge to our proposed strategy stems from the fact that, without taking into consideration the length of the spatial domain a, one cannot obtain non-trivial dynamics on the so-called blow-up locus that is given by $\bar{r} = 0$. To overcome that challenge, we could include a as an auxiliary variable in the quasi-homogeneous blow-up transformation in (1.50) by writing $a = \bar{r}^\eta \bar{a}$, which is the approach taken in [19]. That approach, however, has the disadvantage that the resulting vector fields are not even continuous for $a = 0$, as the exponent η is negative.

A key novelty here, in comparison to [19], is that we adopt an alternative approach by defining a new constant

$$a = A\varepsilon^p, \tag{1.52}$$

with $p \in \mathbb{R}$ to be determined, which allows us to obtain non-trivial dynamics for $\bar{r} = 0$ without the conceptual difficulties encountered in [19]. A transformation such as this changes the fast-slow structure of the problem but is in the spirit of geometric singular perturbation theory; the dynamics in the singular limit $\varepsilon \to 0$ of the new problem organise the dynamics for $\varepsilon > 0$ and for these strictly positive values of ε (1.52) is simply a rescaling of the perturbation parameter. More generally, the question of whether it is more appropriate to rescale a variable before applying the blowup transformation or including it in the system as a dummy variable is commonplace when working with GSPT and the answer depends on the particular problem.

Regardless of the approach used, it appears that some kind of rescaling of the domain in (1.2) is necessary to perform a successful geometric desingularisation, which is an intrinsic consequence of the Galerkin system in (1.28) originating from the discretisation of a system of PDEs and the origin in (1.28) not being a fully degenerate steady state. In sum, after substituting (1.52) into (1.28), we hence obtain

$$u_1' = -v_1 + 2^{-1/2}u_1^2 + 2^{-1/2}\sum_{j=2}^{k_0} u_j^2 + H_1^u, \tag{1.53a}$$

$$v_1' = -2^{1/2}\varepsilon, \tag{1.53b}$$

$$u_k' = \frac{1}{4A^2} b_k \varepsilon^{-2p} u_k - v_k + 2^{1/2} u_1 u_k + \sum_{i,j=2}^{k_0} \eta_{i,j}^k u_i u_j + H_k^u, \tag{1.53c}$$

$$v_k' = \frac{1}{4A^2} b_k \varepsilon^{-2p+1} v_k + \varepsilon H_k^v, \tag{1.53d}$$

$$\varepsilon' = 0. \tag{1.53e}$$

Remark 1.28. The ε-dependent rescaling of the domain for (1.2) through (1.52) changes the fast-slow structure of the original system in (1.28). While the two systems are equivalent for non-zero ε, singular objects such as steady states or manifolds for (1.53) in blowup space no longer correspond directly to singular objects from the layer and reduced problems for (1.28). In addition, the origin of (1.53) is now a *fully degenerate* steady state.

To determine $\alpha_k, \beta_k, \zeta, \delta$ in (1.50), a heuristic is to consider the rescaling chart $K_2 : \{\bar{\varepsilon} = 1\}$ with coordinates given by

$$u_k = r_2^{\alpha_k} u_{k,2}, \quad v_k = r_2^{\beta_k} v_{k,2}, \quad \varepsilon = r_2^{\zeta}, \quad \text{for } k = 1,2,\ldots,k_0$$

and replace into (1.53):

$$\begin{aligned}
\partial_t u_{1,2} &= -r_2^{\beta_1-\alpha_1} v_{1,2} + 2^{-1/2} r_2^{\alpha_1} u_{1,2}^2 + 2^{-1/2} \sum_{j=2}^{k_0} r_2^{2\alpha_j-\alpha_1} u_{j,2}^2 + \mathcal{O}(\ldots), && (1.54a)\\
\partial_t v_{1,2} &= -2^{1/2} r_2^{\zeta-\beta_1} + \mathcal{O}(\ldots), && (1.54b)\\
\partial_t u_{k,2} &= \frac{1}{4A^2} b_k r_2^{-2p\zeta} u_{k,2} - r_2^{\beta_k-\alpha_k} v_{k,2} + 2^{1/2} r_2^{\alpha_1} u_{1,2} u_{k,2} && (1.54c)\\
&\quad + \sum_{i,j=2}^{k_0} \eta_{i,j}^k r_2^{\alpha_i+\alpha_j-\alpha_k} u_{i,2} u_{j,2} + \mathcal{O}(\ldots) &&\\
\partial_t v_{k,2} &= \frac{1}{4A^2} b_k r_2^{\zeta-2p\zeta} v_{k,2} + \mathcal{O}(\ldots), && (1.54d)\\
\partial_t r_2 &= 0. && (1.54e)
\end{aligned}$$

Balancing powers of r_2 on the right hand side, we see that the following relations must hold:

$$\begin{aligned}
\beta_1 &= 2\alpha_1, && (1.55a)\\
\alpha_k &= \alpha_1 \quad \text{for } 2 \le k \le k_0, && (1.55b)\\
\zeta - \beta_1 &= \alpha_1, && (1.55c)\\
-2p\zeta &\ge \alpha_1, && (1.55d)\\
\beta_j &= 2\alpha_1 \quad \text{for } 2 \le j \le k_0, && (1.55e)\\
\zeta - 2p\zeta &\ge \alpha_1. && (1.55f)
\end{aligned}$$

We see from the first three equations above that the consecutive ratios $\alpha_k : \beta_k : \zeta$ between the exponents α_k, β_k, and ζ must be $1:2:3$, as in the finite-dimensional case [34]. The smallest positive integers and the resulting power p that satisfy these relations are

$$\alpha_k = 1, \quad \beta_k = 2, \quad \zeta = 3, \quad \text{and} \quad p = -\frac{1}{6}. \tag{1.56}$$

Remark 1.29. The choice $p = -\frac{1}{6}$ is the unique one that leaves no factor of r_i after desingularisation in the resulting equations for $u_{k,i}$ in chart K_i $(i = 1,2,3)$, where one also requires equality in (1.55d), making use of the relation $3\alpha_1 = \zeta$. Furthermore, note that the weights in (1.56) are consistent with the scaling obtained from a “desingularisation” of the system of PDEs in (1.2); see Section 1.6 for details.

For future reference, we also state the changes of coordinates between charts K_1, K_2, and K_3, as follows.

Lemma 1.30. *The change of coordinates κ_{12} between charts K_1 and K_2 and its inverse κ_{21} are given by*

$$\kappa_{12}: \; u_{1,2} = \varepsilon_1^{-1/3} u_{1,1}, \; v_{1,2} = \varepsilon_1^{-2/3}, \; u_{k,2} = \varepsilon_1^{-1/3} u_{k,1}, \; v_{k,2} = \varepsilon_1^{-2/3} v_{k,1}, \; \textit{and}\; r_2 = \varepsilon_1^{1/3} r_1; \tag{1.57}$$

$$\kappa_{12}^{-1}: \; u_{1,1} = v_{1,2}^{-1/2} u_{1,2}, \; r_1 = v_{1,2}^{1/2} r_2, \; u_{k,1} = v_{1,2}^{-1/2} u_{k,2}, \; v_{k,1} = v_{1,2}^{-1} v_{k,2}, \; \textit{and}\; \varepsilon_1 = v_{1,2}^{-3/2}; \tag{1.58}$$

between charts K_2 and K_3, we have the change of coordinates

$$\kappa_{23}: \; r_3 = u_{1,2} r_2, \; v_{1,3} = u_{1,2}^{-2} v_{1,2}, \; u_{k,3} = u_{1,2}^{-1} u_{k,2}, \; v_{k,3} = u_{1,2}^{-2} v_{k,2}, \; \textit{and}\; \varepsilon_3 = u_{1,2}^{-3}. \tag{1.59}$$

Proof. Direct calculation. □

1.5.1 Chart K_1

The coordinate chart K_1 is formally defined by $\bar{v}_1 = 1$; expressed in the coordinates of that chart, the blow-up transformation in (1.50) reads

$$u_1 = r_1 u_{1,1}, \quad v_1 = r_1^2, \quad u_k = r_1 u_{k,1}, \quad v_k = r_1^2 v_{k,1}, \quad \text{and} \quad \varepsilon = r_1^3 \varepsilon_1.$$

With that transformation, the system in (1.53) becomes, after desingularisation of the resulting vector field by a factor of r_1,

$$u_{1,1}' = F_1 u_{1,1} - 1 + 2^{-1/2} u_{1,1}^2 + 2^{-1/2} \sum_{j=2}^{k_0} u_{j,1}^2 + H_{1,1}^u, \tag{1.60a}$$

$$r_1' = -r_1 F_1, \tag{1.60b}$$

$$u_{k,1}' = F_1 u_{k,1} + \frac{b_k}{4A^2} \varepsilon_1^{1/3} u_{k,1} - v_{k,1} + 2^{1/2} u_{1,1} u_{k,1} + \sum_{i,j=2}^{k_0} \eta_{i,j}^k u_{i,1} u_{j,1} + H_{k,1}^u, \tag{1.60c}$$

$$v_{k,1}' = 2F_1 v_{k,1} + \frac{b_k}{4A^2} r_1^3 \varepsilon_1^{4/3} v_{k,1} + \varepsilon_1 H_{k,1}^v, \tag{1.60d}$$

$$\varepsilon_1' = 3F_1 \varepsilon_1, \tag{1.60e}$$

where

$$F_1 = F_1(\varepsilon_1) = 2^{-1/2} \varepsilon_1,$$

as well as

$$\begin{aligned}
H_{1,1}^u &= \mathcal{O}\left(r_1\varepsilon_1, r_1^2, r_1^2 v_{j,1}^2, r_1 u_{1,1}, r_1 u_{j,1} v_{j,1}, r_1 u_{1,1} u_{j,1}^2, r_1 u_{1,1} u_{j,1}^2, r_1 u_{j,1} u_{i,1} u_{l,1}\right), \\
H_{k,1}^u &= \mathcal{O}\left(r_1^2 v_{k,1}, r_1^2 v_{i,1} v_{j,1}, r_1 u_{1,1} v_{k,1},\right. \\
&\qquad \left. r_1 u_{k,1}, r_1 u_{i,1} v_{j,1}, r_1 u_{1,1}^2 u_{k,1}, r_1 u_{1,1} u_{i,1} u_{j,1}, r_1 u_{j,1} u_{i,1} u_{l,1}\right), \\
H_{k,1}^v &= \mathcal{O}\left(r_1^4 v_{k,1}, r_1^4 v_{i,1} v_{j,1}\right),
\end{aligned}$$

for $2 \le i, j, l \le k_0$ and $2 \le k \le k_0$. Due to the presence of fractional powers of ε_1 in Equations (1.60c) and (1.60d) for $u_{k,1}$ and $v_{k,1}$, respectively, the corresponding flow will not even be C^1-smooth in ε_1. Hence, we rewrite (1.60) in terms of $\varepsilon_1^{1/3}$, which gives

$$u_{1,1}' = F_1 u_{1,1} - 1 + 2^{-1/2} u_{1,1}^2 + 2^{-1/2} \sum_{j=2}^{k_0} u_{j,1}^2 + H_{1,1}^u, \tag{1.61a}$$

$$r_1' = -r_1 F_1, \tag{1.61b}$$

$$u_{k,1}' = F_1 u_{k,1} + \frac{b_k}{4A^2} \left(\varepsilon_1^{1/3}\right) u_{k,1} - v_{k,1} + 2^{-1/2} u_{1,1} u_{k,1} + \sum_{i,j=2}^{k_0} \eta_{i,j}^k u_{i,1} u_{j,1} + H_{k,1}^u, \tag{1.61c}$$

$$v_{k,1}' = 2F_1 v_{k,1} + \frac{b_k}{4A^2} r_1^3 \left(\varepsilon_1^{1/3}\right)^4 v_{k,1} + (\varepsilon_1^{1/3})^3 H_{k,1}^v, \tag{1.61d}$$

$$\left(\varepsilon_1^{1/3}\right)' = F_1 \left(\varepsilon_1^{1/3}\right). \tag{1.61e}$$

Clearly, the flow of the transformed system in (1.61) will be smooth with respect to $\varepsilon_1^{1/3}$; in the following, we will refer to that system when a higher degree of smoothness is required.

Equation (1.61) admits the two principal steady states

$$p_a^{k_0} := (-2^{1/4}, 0, 0, 0, 0) \quad \text{and} \quad p_r^{k_0} := (2^{1/4}, 0, 0, 0, 0). \tag{1.62}$$

Lemma 1.31. *The point $p_a^{k_0}$ is a partially hyperbolic steady state of Equation (1.61), with the following eigenvalues and eigenvectors in the corresponding linearisation:*

- *the simple eigenvalue $-2^{3/4}$ with eigenvector $(1, 0, \ldots, 0)$, corresponding to $u_{1,1}$;*
- *the eigenvalue $-2^{3/4}$ with multiplicity $k_0 - 1$ and eigenvectors $(0, 0, 0, \ldots, 1, \ldots, 0)$, where non-zero entries appear at the $(k+2)$-th position, corresponding to $u_{k,1}$ with $2 \leq k \leq k_0$;*
- *the eigenvalue 0 with multiplicity $k_0 + 1$, corresponding to r_1, $v_{k,1}, 2 \leq k \leq k_0$ and $\varepsilon_1^{1/3}$.*

Proof. Direct calculation. □

To describe the transition through chart K_1, i.e., to approximate the corresponding transition map, we define the following sections for the flow of (1.60):

$$\Sigma_{1,k_0}^{\text{in}} := \{(u_{1,1}, r_1, u_{k,1}, v_{k,1}, \varepsilon_1) \, : \, r_1 = \rho\} \quad \text{and} \quad \Sigma_{1,k_0}^{\text{out}} := \{(u_{1,1}, r_1, u_{k,1}, v_{k,1}, \varepsilon_1) \, : \, \varepsilon_1 = \delta\}, \tag{1.63}$$

for some $\delta > 0$ sufficiently small. Next, we need to determine the transition time between $\Sigma_{1,k_0}^{\text{in}}$ and $\Sigma_{1,k_0}^{\text{out}}$, which will allow us to derive estimates for the corresponding orbits as they pass through chart K_1.

Lemma 1.32. *The transition time between the sections $\Sigma_{1,k_0}^{\text{in}}$ and $\Sigma_{1,k_0}^{\text{out}}$ under the flow of (1.60) is given by*

$$T_1 = \frac{\sqrt{2}}{3} \left(\frac{1}{\varepsilon_1(0)} - \frac{1}{\delta} \right). \tag{1.64}$$

Proof. The explicit solution of Equation (1.60e) for ε_1 reads

$$\varepsilon_1(t) = \frac{2\varepsilon_1(0)}{2 - 3\sqrt{2}\varepsilon_1(0)t}, \tag{1.65}$$

where $\varepsilon_1(0)$ denotes an appropriately chosen initial value for ε_1 in $\Sigma^{\text{in}}_{1,k_0}$. Solving the equation $\varepsilon_1(T_1) = \delta$ for T_1 results in (1.64), as stated. (Note that the denominator in (1.65) remains strictly positive for all $t \in [0, T_1]$.) □

Remark 1.33. We refer to the time variable by t throughout for simplicity of notation, even though we consider different systems in the three coordinate charts K_j ($j = 1,2,3$), as well as multiple parametrisations of the same system in some cases.

To give a more complete description of the geometry – in particular of the steady state structure – we proceed as follows. Setting $r_1 = 0 = \varepsilon_1$ in (1.60), we find the singular system

$$u'_{1,1} = -1 + 2^{-1/2}u^2_{1,1} + 2^{-1/2}\sum_{j=2}^{k_0} u^2_{j,1}, \tag{1.66a}$$

$$r'_1 = 0, \tag{1.66b}$$

$$u'_{k,1} = -v_{k,1} + 2^{1/2}u_{1,1}u_{k,1} + \sum_{i,j=2}^{k_0} u_{i,1}u_{j,1}, \tag{1.66c}$$

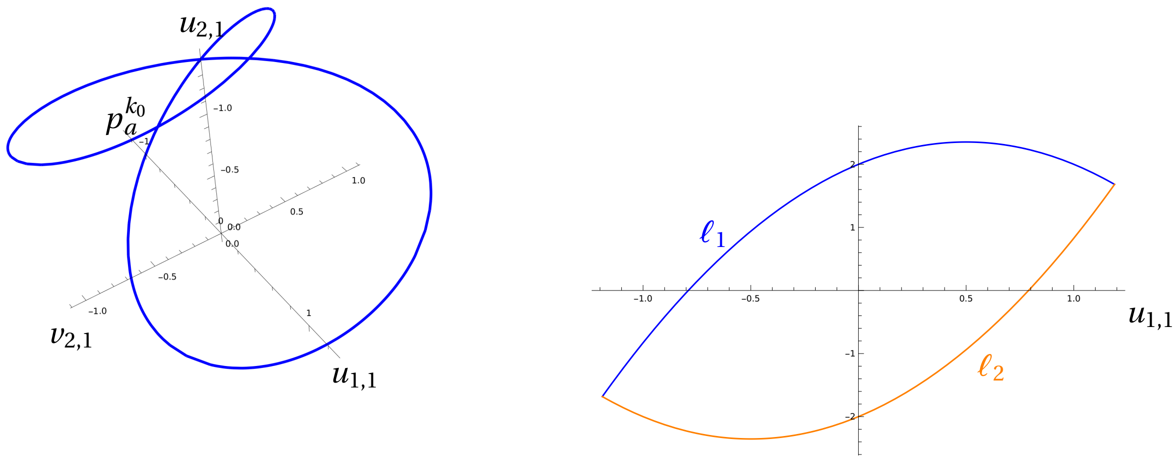
$$v'_{k,1} = 0, \tag{1.66d}$$

$$\varepsilon'_1 = 0, \tag{1.66e}$$

from which we see that the hyperplanes $\{r_1 = 0\}$ and $\{\varepsilon_1 = 0\}$ are invariant, as is their intersection. An application of the implicit function theorem shows that lines of steady states emanate from $p^{k_0}_a$ and $p^{k_0}_r$, respectively, for $u_{1,1}$ close to $\mp 2^{1/4}$ and $u_{k,1}$ and $v_{k,1}$ small ($2 \le k \le k_0$). Locally, around $p^{k_0}_a$, these steady states will inherit the stability of $p^{k_0}_a$, which we will make use of in the estimates in the following subsection. For $k_0 = 2$, the geometry is exemplified in Figures 1.5a and 1.5b, in which case p^2_a and p^2_r are connected by curves of steady states which can be calculated explicitly from (1.66); see Figure 1.5a. The linearisation around those states has one zero eigenvalue and two non-trivial eigenvalues ℓ_1 and ℓ_2 which depend on the $u_{1,1}$-coordinate only; these eigenvalues are plotted in Figure 1.5b.

The geometry for general k_0 will be similar, in that $p^{k_0}_a$ and $p^{k_0}_r$ will not be isolated, with steady states lying in the plane $\{r_1 = 0 = \varepsilon_1\}$ that are neutral in the $v_{k,1}$-directions and of varying stability in $u_{1,1}$ and $u_{k,1}$, for $2 \le k \le k_0$. States that are close to the point $p^{k_0}_a$ will be stable in the latter directions, while those close to $p^{k_0}_r$ will be unstable in the same directions; in between, there will be steady states of saddle type. These statements are a direct consequence of the implicit function theorem, applied to the vector field in (1.66). It is unclear whether a curve of steady states that connects $p^{k_0}_a$ and $p^{k_0}_r$ will exist for general k_0, as is the case for $k_0 = 2$.

Furthermore, lines of equilibria are found emanating from each steady state in $\{r_1 = 0 = \varepsilon_1 = 0\}$, as can again be seen from the implicit function theorem. These lines locally inherit the stability of the corresponding steady state they are based on.



(a) Steady states of (1.66) for $k_0 = 2$..

(b) The two non-trivial eigenvalues ℓ_1 and ℓ_2.

Figure 1.5: Steady state structure of (1.66) for $k_0 = 2$. (a) The principal steady states p_a^2 and p_r^2 are connected by a pair of symmetric curves of steady states that are parametrised by $u_{1,1}$. The closed curve of steady states corresponds to the intersection of the full critical manifold with the singular surface $\{r_1 = 0\}$ (b) The two non-trivial eigenvalues ℓ_1 and ℓ_2 of the linearisation about these steady states are plotted against $u_{1,1}$.

Remark 1.34. Typically, steady states in the equivalent of the subspace $\{r_1 = 0 = \varepsilon_1\}$ after blow-up, and the lines of states that emanate from them, can be viewed as intersections of critical and slow manifolds, respectively, with the blow-up space [34]. However, that is not the case here, as the rescaling of the spatial domain through ε with the introduction of A in (1.52) alters the fast-slow structure of the original Equation (1.28).

The existence of non-hyperbolic steady states near $p_a^{k_0}$ that are attracting in $u_{1,1}$ and $u_{k,1}$ $(k = 2, \dots, k_0)$ implies the following lemma.

Lemma 1.35. *For sufficiently small* ρ, δ, $C^{\text{in}}_{u_{1,1}}$, $C^{\text{in}}_{u_{k,1}}$, *and* $C^{\text{in}}_{v_{k,1}}$, *there exists an attracting,* (k_0+2)*-dimensional centre manifold* $M_{k_0,1}$ *at* $p_a^{k_0}$ *in* (1.61)*. The manifold* $M_{k_0,1}$ *is given as a graph over* $\left(u_{1,1}, r_1, v_{k,1}, \varepsilon_1^{1/3}\right)$*, where* $2 \le k \le k_0$*. In particular, for initial conditions close to* $p_a^{k_0}$*, solutions of* (1.61) *satisfy* $u_{1,1}(t) < 0$ *for* $t \in [0, T_1]$.

Proof. The statements follow from centre manifold theory and Lemma 1.31. □

The centre manifold argument in Lemma 1.35 implies that if $u_{1,1}$ is close to $-2^{1/4}$ initially, then it will remain close throughout the transition through chart K_1 – in particular, $u_{1,1}$ will remain negative. To obtain corresponding estimates for the remaining variables $u_{k,1}$ and $v_{k,1}$, with $k = 2, \dots, k_0$, we combine the classical variation of constants formula with a fixed point argument, as follows.

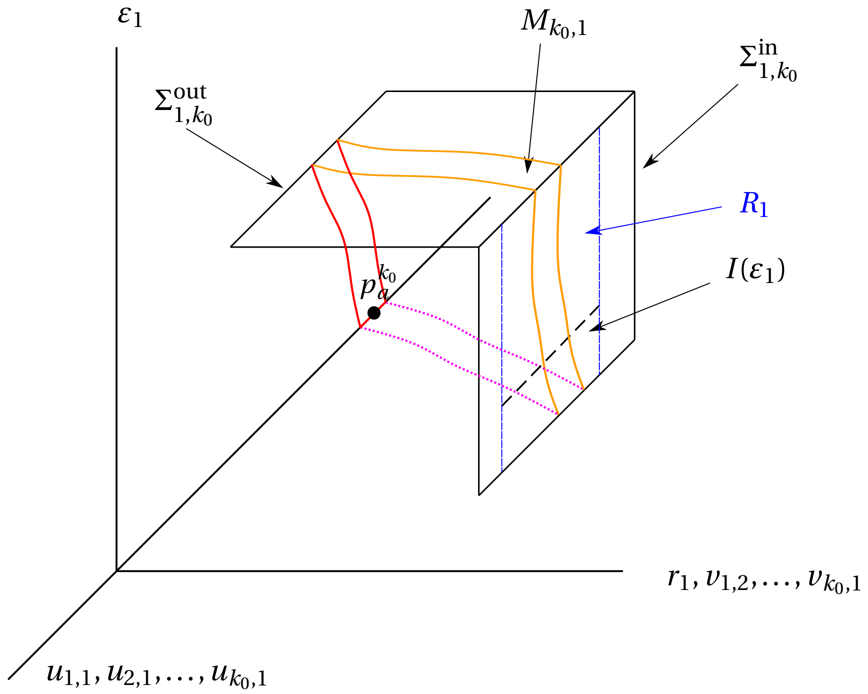


Figure 1.6: The dynamics in K_1 are organised around the attracting centre manifold $M_{k_0,1}$, which is anchored in a curve of steady states in the subspace $\{r_1 = 0 = \mu_1\}$, one of which is $p_a^{k_0}$. The transition map Π_1 is defined on the subset $R_1 \subset \Sigma^{\text{in}}_{1,k_0}$ around the intersection $\Sigma^{\text{in}}_{1,k_0} \cap M_{k_0,1}$; slices of R_1 with ε_1 constant, denoted by $I(\varepsilon_1)$, will be mapped to slices with constant ε_3 in chart K_3.

Lemma 1.36. *For $2 \le k \le k_0$ and $u_k(0)$ and $v_k(0)$ as defined in (1.31), solutions of Equation (1.60) satisfy the estimates*

$$\left|u_{k,1}(t)\right| \le \frac{1}{\rho}|u_k(0)| + \frac{8a^2\rho}{|b_k|}\left[\sigma_u + \sigma_v\varepsilon_1(0)^{2/3}\rho^2\delta^{2/3}\left(1+\frac{8a^2}{|b_k|}\right)\right] \tag{1.67}$$

and

$$\left|v_{k,1}(t)\right| \le \frac{\delta^{2/3}}{\varepsilon_1(0)^{2/3}\rho^2}\left|v_k(0)\right| + \varepsilon_1(0)^{2/3}\delta^{2/3}\frac{8a^2\rho^2}{|b_k|}\sigma_v \le \varepsilon_1(0)^{2/3}\rho^2\delta^{2/3}\left(C_{k,v_0} + \frac{8a^2}{|b_k|}\sigma_v\right) \tag{1.68}$$

for all $t \in [0, T_1]$ and some $\kappa \le \sigma_u, \sigma_v < 1$, where T_1 is the transition time determined in (1.64) and $\kappa > 0$ is as in (1.30).

Proof. We first derive the estimates for $v_{k,1}$. Application of the variation of constants formula to (1.60d) yields

$$v_{k,1}(t) = \exp\left(\int_0^t V_{k,1}(s)\,ds\right)v_{k,1}(0) + \int_0^t \exp\left(\int_s^t V_{k,1}(\tau)\,d\tau\right)\varepsilon_1(s)H^v_{k,1}\,ds, \tag{1.69}$$

where $V_{k,1}(s) = 2^{1/2}\varepsilon_1(s) + \frac{b_k}{4A^2}r_1^3(s)\varepsilon_1^{4/3}(s)$. Equation (1.60b) can be solved explicitly for r_1 to give

$$r_1(t) = 2^{-1/3}\rho\left(2 - 3\sqrt{2}\varepsilon_1(0)\,t\right)^{1/3}, \tag{1.70}$$

where $\varepsilon_1(0)$ denotes the initial value for ε_1 and $r_1(0) = \rho$. Note that, due to $b_k < 0$, the second term in $V_{k,1}(s)$ is negative for all $s \in [0, T_1]$. Combining the above expression with the explicit solutions for $\varepsilon_1(t)$ and $r_1(t)$ in (1.65) and (1.70), respectively, then implies

$$\begin{aligned} |v_{k,1}(t)| \le \frac{2}{3}\exp\Big(\int_0^t \frac{\varepsilon_1(0)}{\phi_1(s)}\,ds\Big)|v_{k,1}(0)| \\ +\Big(\frac{\sqrt{2}}{3}\Big)^{\frac{1}{3}}\rho^2\frac{e^{-\alpha\phi_1(t)^{2/3}}}{\phi_1(t)^{2/3}}\int_0^t \varepsilon_1(0)\phi_1(s)^{1/3}e^{\alpha\phi_1(s)^{2/3}}|\widetilde{H}^v_{k,1}(s)|\,ds, \end{aligned} \tag{1.71}$$

where $\phi_1(s) = \sqrt{2}/3 - \varepsilon_1(0)s$, $\alpha = (3/\sqrt{2})^{2/3}b_k\rho^2/(4a^2\sqrt{2})$, and $H^v_{k,1}(t) = r_1(t)^2\widetilde{H}^v_{k,1}(t)$. Evaluating the integrals in (1.71), we obtain

$$\big|v_{k,1}(t)\big| \le \frac{\delta^{2/3}}{\varepsilon_1(0)^{2/3}}\Big(|v_{k,1}(0)| + \frac{8a^2}{|b_k|}\sup_{[0,T_1]}|\widetilde{H}^v_{k,1}(t)|\Big) \qquad \text{for all } t \in [0, T_1]. \tag{1.72}$$

Recall that the term $\widetilde{H}^v_{k,1}$ is at least quadratic terms in $v_{k,1}$, with $k = 2, \dots k_0$. To estimate $u_{k,1}$, we first rewrite (1.60c) in the form

$$u'_{k,1} = \Big(F_1 + \frac{b_k}{4A^2}\varepsilon_1^{1/3} + 2^{1/2}u_{1,1}\Big)u_{k,1} - v_{k,1} + M_k(u_{2,1}, \dots, u_{k_0,1}) + H^u_{k,1}$$

for $2 \le k \le k_0$, where

$$M_k(u_{2,1}, \dots, u_{k_0,1}) := \sum_{i,j=2}^{k_0}\eta^k_{i,j}u_{i,1}u_{j,1}.$$

Application of the variation of constants formula yields

$$\begin{aligned} u_{k,1}(t) = \exp\Big(\int_0^t U_{k,1}(s)\,ds\Big)u_{k,1}(0) - \int_0^t \exp\Big(\int_s^t U_{k,1}(\tau)\,d\tau\Big)v_{k,1}(s)\,ds \\ + \int_0^t \exp\Big(\int_s^t U_{k,1}(\tau)\,d\tau\Big)\Big(M(u_{2,1}(s), \dots, u_{k_0,1}(s)) + H^u_{k,1}\Big)ds, \end{aligned} \tag{1.73}$$

where $U_{k,1}(s) := 2^{-1/2}\varepsilon_1(s) + \frac{b_k}{4A^2}\varepsilon_1^{1/3}(s) + 2^{1/2}u_{1,1}(s)$.

Due to Lemma 1.35, we have $u_{1,1}(s) < 0$ for $s \in [0, T_1]$; hence, in the following estimates, we replace $U_{k,1}(s)$ with $\widetilde{U}_{k,1}(s) := 2^{-1/2}\varepsilon_1(s) + \frac{b_k}{4A^2}\varepsilon_1^{1/3}(s)$, as $\exp\big(\int_s^t U_{k,1}(\tau)\,d\tau\big) \le \exp\big(\int_s^t \widetilde{U}_{k,1}(\tau)\,d\tau\big)$ for all $0 \le s < t \le T_1$. Direct integration gives

$$\begin{aligned} 0 \le \mathscr{I}_1(t) = \exp\Big(\int_0^t \widetilde{U}_{k,1}(s)\,ds\Big) \\ = \frac{1}{(1-(3/\sqrt{2})\varepsilon_1(0)\,t)^{1/3}}\exp\Big(\frac{b_k}{4A^2\sqrt{2}}\Big[\Big(\frac{1}{\varepsilon_1(0)}\Big)^{\frac{2}{3}} - \Big(\frac{1}{\varepsilon_1(0)} - \frac{3}{\sqrt{2}}t\Big)^{\frac{2}{3}}\Big]\Big) \le 1, \end{aligned} \tag{1.74}$$

since $\mathscr{I}_1(t)$ is a non-increasing function for $\delta \le \pi^3 2^{-5/4}/(\rho^{3/2}\varepsilon_1(0)^{1/2}) = \pi^3 2^{-5/4}/\varepsilon^{1/2}$, and

$$\mathscr{I}_1(T_1) = \frac{\delta^{1/3}}{\varepsilon_1(0)^{1/3}}\exp\Big(\frac{b_k}{4A^2\sqrt{2}}\Big[\Big(\frac{1}{\varepsilon_1(0)}\Big)^{\frac{2}{3}} - \Big(\frac{1}{\delta}\Big)^{\frac{2}{3}}\Big]\Big) \le \exp\Big(\frac{b_k}{8A^2\sqrt{2}}\Big(1-\frac{1}{\alpha}\Big)\Big(\frac{1}{\varepsilon_1(0)}\Big)^{\frac{2}{3}}\Big)$$

for $\delta \geq \alpha\varepsilon_1(0) > 0$ with $\alpha \geq 1$. Next, we have that

$$0 \leq \mathscr{I}_2(t) = \int_0^t \exp\left(\int_s^t \widetilde{U}_{k,1}(\tau)d\tau\right)ds$$
$$= \varepsilon_1(t)^{\frac{1}{3}}\frac{4A^2}{|b_k|}\left[\left(\frac{1}{\varepsilon_1(0)}-\frac{3}{\sqrt{2}}t\right)^{2/3}-\left(\frac{1}{\varepsilon_1(0)}\right)^{2/3}\exp\left(\frac{1}{\sqrt{2}}\frac{b_k}{4A^2}\left[\left(\frac{1}{\varepsilon_1(0)}\right)^{\frac{2}{3}}-\left(\frac{1}{\varepsilon_1(0)}-\frac{3}{\sqrt{2}}t\right)^{2/3}\right]\right)\right]$$
$$+\varepsilon_1(t)^{\frac{1}{3}}\frac{16A^4\sqrt{2}}{|b_k|^2}\left[1-\exp\left(\frac{1}{\sqrt{2}}\frac{b_k}{4A^2}\left[\left(\frac{1}{\varepsilon_1(0)}\right)^{\frac{2}{3}}-\left(\frac{1}{\varepsilon_1(0)}-\frac{3}{\sqrt{2}}t\right)^{\frac{2}{3}}\right]\right)\right]$$
$$\leq \frac{8\rho a^2}{|b_k|},$$

where we have used $A^2 = \varepsilon^{1/3}a^2 = \rho\varepsilon_1(0)^{1/3}a^2$, and

$$\mathscr{I}_2(T_1) = \delta^{-1/3}\frac{4A^2}{|b_k|}\left[1-\left(\frac{\delta}{\varepsilon_1(0)}\right)^{2/3}\exp\left(\frac{1}{\sqrt{2}}\frac{b_k}{4A^2}\left[\left(\frac{1}{\varepsilon_1(0)}\right)^{2/3}-\frac{1}{\delta^{2/3}}\right]\right)\right]$$
$$+\delta^{1/3}\frac{16A^4\sqrt{2}}{|b_k|^2}\left[1-\exp\left(\frac{1}{\sqrt{2}}\frac{b_k}{4A^2}\left[\left(\frac{1}{\varepsilon_1(0)}\right)^{\frac{2}{3}}-\frac{1}{\delta^{2/3}}\right]\right)\right] \leq \frac{1}{\delta^{1/3}}\frac{4A^2}{|b_k|}\left[1+\frac{4A^2\sqrt{2}}{|b_k|}\right],$$

where again $0 < \alpha\varepsilon_1(0) \leq \delta < 1$ with $\alpha \geq 1$.

To complete the estimates, we shall use a fixed point argument. To that end, we define the set

$$\mathscr{B}_1 = \Big\{(\tilde{u}_{2,1},\dots,\tilde{u}_{k_0,1},\tilde{v}_{2,1},\dots,\tilde{v}_{k_0,1}) : \tilde{u}_{k,1},\tilde{v}_{k,1} \in C([0,T]), 2 \leq k \leq k_0,$$
$$\text{with } \sup_{[0,T_1]}|\tilde{u}_{k,1}(t)| \leq C_{k,u}, \sup_{[0,T_1]}|\tilde{v}_{k,1}(t)| \leq C_{k,v},$$
$$\text{and } \sum_{k=2}^{k_0} C_{k,u}^2 \leq \tilde{\sigma}_u, \sum_{k=2}^{k_0} C_{k,v}^2 \leq \tilde{\sigma}_v\varepsilon_1(0)^{4/3}\Big\},$$

where $\tilde{\sigma}_u, \tilde{\sigma}_v \leq 1$.

Considering in (1.69) and (1.73) $M_k(\tilde{u}_{2,1},\dots,\tilde{u}_{k_0,1})$ and

$$H^l_{k,1} = H^l_{k,1}(\tilde{u}_{2,1},\dots,\tilde{u}_{k_0,1},\tilde{v}_{2,1},\dots,\tilde{v}_{k_0,1})$$

with $l = u, v$, for $(\tilde{u}_{2,1},\dots,\tilde{u}_{k_0,1},\tilde{v}_{k,1},\dots,\tilde{v}_{k_0,1}) \in \mathscr{B}_1$, we obtain a map $\mathscr{N}_1$ given by $\mathscr{N}_1(\tilde{u}_{2,1},\dots,\tilde{u}_{k_0,1},\tilde{v}_{2,1},\dots,\tilde{v}_{k_0,1}) = (u_{2,1},\dots,u_{k_0,1},v_{2,1},\dots,v_{k_0,1})$; in particular, solutions of (1.69) and (1.73) correspond to fixed points of $\mathscr{N}_1$.

We shall show that $\mathscr{N}_1 : \mathscr{B}_1 \to \mathscr{B}_1$. Our assumptions on the initial conditions, together with (1.72), yield

$$\left|v_{k,1}(t)\right| \leq \delta^{2/3}\varepsilon_1(0)^{2/3}\left(\rho^2 C_{k,v_0} + \frac{8a^2\rho^2}{|b_k|}\sigma_v\right) \qquad \text{for all } t \in [0,T_1], \tag{1.75}$$

where we used the estimate $|\widetilde{H}^v_{k,1}(t)| \leq C_1\rho^2\sum_{k=2}^{k_0}|\tilde{v}_{k,1}|^2 \leq \rho^2 C_2\varepsilon_1(0)^{4/3}\tilde{\sigma}_v \leq \rho^2\varepsilon_1(0)^{4/3}\sigma_v$.

Then,

$$|u_{k,1}(t)| \le |u_{k,1}(0)| + \frac{8a^2\rho}{|b_k|}\big(C_3\varepsilon_1(0)^{2/3} + \sigma_u\big) \qquad \text{for all } t \in [0, T_1],$$

where $|M_k + H^u_{k,1}| \le C_4 \sum_{k=2}^{k_0} |\tilde{u}_k|^2 \le C_5 \tilde{\sigma}_u = \sigma_u$.

Thus, for $0 < \rho < 1$ and $0 < \sigma_u, \sigma_v < 1$, we obtain that $\mathcal{N}_1 : \mathcal{B}_1 \to \mathcal{B}_1$, which implies the estimates in (1.67) and (1.68). □

Remark 1.37. Note that if $H^v = 0$, then it is sufficient to consider $|v_k(0)| \le C_{k,v_0}\varepsilon^{1/2}$ and $|u_k(0)| \le C_{k,u_0}$. In the case of more general higher-order terms

$$H^v_k = \mathcal{O}(u_i u_j, v_i v_j, v_1 v_k, v_i v_j)$$

for $i, j = 2, \ldots, k_0$, we would have to assume that $|u_k(0)| \le C_{k,u_0}\varepsilon^{2/3}$. Then, in the definition of $\mathcal{B}_1$, we would consider $\sum_{k=2}^{k_0} C^2_{k,u} \le \varepsilon^{4/3}\tilde{\sigma}_u$, which would imply

$$|u_{k,1}(t)| \le \varepsilon_1(0)^{2/3}\rho^2 C_{k,u_0} + \varepsilon_1(0)^{2/3}\frac{8a^2\rho}{|b_k|}\big(\sigma_u + \sigma_v\big)$$

for $u_{k,1}$.

Given the above estimates, the transition map Π_1 in chart K_1 will be defined on the set $R_1 \subset \Sigma^{\text{in}}_{1,k_0}$, see Figure 1.6, which is given by

$$R_1 := \Big\{(u_{1,1}, r_1, u_{k,1}, v_{k,1}, \varepsilon_1) : |u_{1,1} + 2^{1/4}| \le C^{\text{in}}_{u_{1,1}}, r_1 = \rho, |u_{k,1}| \le C^{\text{in}}_{u_{k,1}}, \\ |v_{k,1}| \le C^{\text{in}}_{v_{k,1}}\varepsilon_1^{4/3}, \text{ and } \varepsilon_1 \in [0, \delta]\Big\}. \quad (1.76)$$

The set R_1 is precisely the set $R^{\text{in}} \subset \Delta^{\text{in}}$, transformed into the coordinates of chart K_1. For $\varepsilon_1 \in [0, \delta]$ fixed, we also define the slices $I(\varepsilon_1) \subset R_1$ as

$$I(\varepsilon_1) := \big\{(u_{1,1}, r_1, u_{k,1}, v_{k,1}, \varepsilon_1) \in R_1 \ : \ \varepsilon_1 \in [0, \delta] \ \text{fixed}\big\}. \quad (1.77)$$

These slices will be useful when combining the transition through chart K_1 with those through charts K_2 and K_3, as $I(\varepsilon_1)$ will be mapped to sets with constant ε_3 in an appropriately defined section $\Sigma^{\text{out}}_{3,k_0}$.

We summarise our findings on the transition through chart K_1, and the corresponding map Π_1, below.

Proposition 1.38. *The transition map* $\Pi_1 : R_1 \to \Sigma^{\text{out}}_{1,k_0}$ *is well-defined. For*

$$(u_{1,1}, \rho, u_{k,1}, v_{k,1}, \varepsilon_1) \in R_1, \quad k = 2, \ldots, k_0,$$

denote

$$\Pi_1(u_{1,1}, \rho, u_{k,1}, v_{k,1}, \varepsilon_1) = (u^{\text{out}}_{1,1}, r^{\text{out}}_1, u^{\text{out}}_{k,1}, v^{\text{out}}_{k,1}, \delta). \quad (1.78)$$

Then, the following estimates hold:

$$|u_{1,1}^{\text{out}} + 2^{1/4}| \le C_{u_{1,1}}^{\text{out}}, \tag{1.79a}$$
$$r_1^{\text{out}} \in [0, \rho], \tag{1.79b}$$
$$|u_{k,1}^{\text{out}}| \le C_{u_{k,1}}^{\text{out}}, \quad \textit{and} \tag{1.79c}$$
$$|v_{k,1}^{\text{out}}| \le C_{v_{k,1}}^{\text{out}} \delta^{2/3}, \tag{1.79d}$$

where $C_{u_{k,1}}^{\text{out}}$*,* $C_{u_{k,1}}^{\text{out}}$*, and* $C_{v_{k,1}}^{\text{out}}$ *are appropriately chosen constants. Furthermore, the restriction* $\Pi_1|I(\varepsilon_1)$ *is a contraction, with rate bounded by* $C\exp(cT_1)$*, where* $C > 0$ *and* $-2^{3/4} < c < 0$.

Proof. The estimates in (1.79c) and (1.79d) follow directly from the definition of R_1 in (1.76) and Lemma 1.36, while (1.79b) is immediate from the observation that $r_1(t)$ is decreasing, by (1.61b). Finally, (1.79a) and the stated contraction property are due to Lemma 1.35 and the existence of the attracting centre manifold $M_{k_0,1}$. □

1.5.2 Chart K_2

As will become apparent, the dynamics of (1.53) in chart K_2 can be seen as a regular perturbation of the planar subsystem for the first two modes $\{u_1, v_1\}$, after transformation to K_2. In particular, for $r_2 = 0$, that subsystem reduces to the well-studied Riccati equation [42]. As the requisite analysis is similar to that in the corresponding rescaling chart for the singularly perturbed planar fold [34], we merely outline it here.

In chart K_2, the blow-up transformation in (1.50) reads

$$u_1 = r_2 u_{1,2}, \quad v_1 = r_2^2 v_{1,2}, \quad u_k = r_2 u_{k,2}, \quad v_k = r_2^2 v_{k,2}, \quad \text{and} \quad \varepsilon = r_2^3.$$

In particular, we note that we hence rescale the variables u_k and v_k for $1 \le k \le k_0$ with powers of $r_2 = \varepsilon^{1/3}$ in this chart, which justifies the terminology.

Substitution of the above transformation into (1.53) gives, after desingularisation with a factor of r_2 from the resulting vector field, the following system:

$$u_{1,2}' = -v_{1,2} + 2^{-1/2} u_{1,2}^2 + 2^{-1/2} \sum_{j=2}^{k_0} u_{j,2}^2 + H_{1,2}^u, \tag{1.80a}$$
$$v_{1,2}' = -2^{1/2}, \tag{1.80b}$$
$$u_{k,2}' = \frac{b_k}{4A^2} u_{k,2} - v_{k,2} + 2^{1/2} u_{1,2} u_{k,2} + \sum_{i,j=2}^{k_0} \eta_{i,j}^k u_{j,2} u_{i,2} + H_{k,2}^u, \tag{1.80c}$$
$$v_{k,2}' = \frac{b_k}{4A^2} r_2^3 v_{k,2} + H_{k,2}^v, \tag{1.80d}$$
$$r_2' = 0 \tag{1.80e}$$

for $2 \le k \le k_0$, where

$$H_{1,2}^u = \mathcal{O}(r_2),$$
$$H_{k,2}^u = \mathcal{O}\left(r_2 u_{1,2} v_{i,2}, r_2 u_{i,2} v_{1,2}, r_2 u_{i,2} u_{j,2}, r_2 u_{1,2}^2 u_{i,1}, r_2 u_{1,2} u_{i,2} u_{j,2}, r_2 u_{i,2} u_{j,2} u_{l,2}\right), \quad \text{and}$$

$H^v_{k,2} = \mathcal{O}\left(r_2^4 v_{1,2} v_{i,2}, r_2^4 v_{i,2} v_{j,2}\right),$

with $2 \le i, j, l \le k_0$.

The plane $\{u_{k,2} = 0 = v_{k,2}, 2 \le k \le k_0\} \cap \{r_2 = 0\}$ is invariant under the flow of Equation (1.80); on that plane, (1.80) reduces to

$$u'_{1,2} = -v_{1,2} + 2^{-1/2} u_{1,2}^2, \tag{1.81a}$$

$$v'_{1,2} = -2^{1/2} \tag{1.81b}$$

with $u_{1,2}, v_{1,2} \in \mathbb{R}$, which is a Riccati equation that corresponds to the one found in [34, Proposition 2.3], up to a rescaling. Correspondingly, we have the following.

Proposition 1.39. *The Riccati equation in (1.81) has the following properties:*

1. *Every orbit has a horizontal asymptote $v_{1,2} = v_{1,2}^{+\infty}$, where $v_{1,2}^{+\infty}$ depends on the orbit, such that $u_{1,2} \to +\infty$ as $v_{1,2}$ approaches $v_{1,2}^{+\infty}$ from above.*

2. *There exists a unique orbit γ_2 which can be parametrised as $(u_{1,2}, s(u_{1,2}))$, with $u_{1,2} \in \mathbb{R}$, which is asymptotic to the left branch of the parabola $\{-v_{1,2} + 2^{-1/2}u_{1,2}^2 = 0\}$ for $u_{1,2} \to -\infty$. The orbit γ_2 has a horizontal asymptote $v_{1,2} = -\Omega_0 < 0$ such that $u_{1,2} \to \infty$ as $v_{1,2}$ approaches $-\Omega_0$ from above, where Ω_0 is a positive constant that is defined as in [34].*

3. *The function $s(u_{1,2})$ has the asymptotic expansions*

$$s(u_{1,2}) = 2^{-1/2} u_{1,2}^2 + \frac{2^{-1/2}}{u_{1,2}} + \mathcal{O}\left(\frac{1}{u_{1,2}^4}\right) \quad \textit{as } u_{1,2} \to -\infty \tag{1.82}$$

and

$$s(u_{1,2}) = -\Omega_0 + \frac{2^{1/2}}{u_{1,2}} + \mathcal{O}\left(\frac{1}{u_{1,2}^3}\right) \quad \textit{as } u_{1,2} \to +\infty. \tag{1.83}$$

4. *All orbits to the right of γ_2 are backward asymptotic to the right branch of the parabola $\{-v_{1,2} + 2^{-1/2}u_{1,2}^2 = 0\}$.*

5. *All orbits to the left of γ_2 have a horizontal asymptote $v_{1,2} = v_{1,2}^{-\infty} > v_{1,2}^{+\infty}$, where $v_{1,2}^{-\infty}$ depends on the orbit, such that $u_{1,2} \to -\infty$ as $v_{1,2}$ approaches $v_{1,2}^{-\infty}$ from above.*

If we transform the orbit γ_2 to chart K_1, we find that

$$\gamma_1 := \kappa_{12}^{-1}(\gamma_2) = \left\{\left(u_{1,2} s(u_{1,2})^{-1/2}, 0, \mathbf{0}, \mathbf{0}, s(u_{1,2})^{-3/2}\right)\right\}, \tag{1.84}$$

where $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^{k_0-1}$.

In fact, expanding (1.84) in a power series as $u_{1,2} \to -\infty$, we find

$$\gamma_1 = \left\{\left(-2^{1/4} + \frac{2^{-3/4}}{u_{1,2}^3} + \mathcal{O}\left(\frac{1}{u_{1,2}^6}\right), 0, \mathbf{0}, \mathbf{0}, -\frac{2^{-3/4}}{u_{1,2}^3} + \mathcal{O}\left(\frac{1}{u_{1,2}^6}\right)\right)\right\}, \tag{1.85}$$

which implies that γ_1 approaches the steady state $p_a^{k_0}$ in chart K_1, tangent to the vector $(-1,0,0,\mathbf{0},\mathbf{0},1)$.

Similarly, for $u_{1,2} > 0$, we can transform γ_2 to the coordinates in chart K_3 via

$$\begin{aligned}\gamma_3 := \kappa_{23}(\gamma_2) &= \left\{\left(0, u_{1,2}^{-2} s(u_{1,2}), 0, 0, u_{1,2}^{-3}\right)\right\} \\ &= \left\{\left(0, -\frac{\Omega_0}{u_{1,2}^2} + \frac{2^{1/2}}{u_{1,2}^3} + \mathcal{O}\left(\frac{1}{u_{1,2}^5}\right), \mathbf{0}, \mathbf{0}, \frac{1}{u_{1,2}^3}\right)\right\} \\ &= \left\{\left(0, -\Omega_0 \varepsilon_3^{2/3} + 2^{1/2}\varepsilon_3 + \mathcal{O}\left(\varepsilon_3^{5/3}\right)\right), \mathbf{0}, \mathbf{0}, \varepsilon_3\right\},\end{aligned} \tag{1.86}$$

which shows that, as $u_{1,2} \to \infty$ – or, equivalently, as $\varepsilon_3 \to 0$ – γ_3 approaches the origin in chart K_3 tangent to the vector $(0,1,\mathbf{0},\mathbf{0},0)$.

To determine the transition map for chart K_2, we first transform the exit section $\Sigma_{1,k_0}^{\text{out}}$ from chart K_1 to the coordinates of chart K_2, applying the change of coordinates κ_{12} in (1.57), which will yield an entry section $\Sigma_{2,k_0}^{\text{in}}$ for the flow in K_2:

$$\Sigma_{2,k_0}^{\text{in}} := \{(u_{1,2}, v_{1,2}, u_{k,2}, v_{k,2}, r_2) : v_{1,2} = \delta^{-2/3}\}.$$

In addition, the orbit γ_2 intersects this section in a single point q_0, so that

$$\gamma_2 \cap \Sigma_{2,k_0}^{\text{in}} = \{q_0\}. \tag{1.87}$$

The coordinates of q_0 satisfy $u_{k,2} = 0 = v_{k,2}$ and $r_2 = 0$. We also define the exit section $\Sigma_{2,k_0}^{\text{out}}$ by

$$\Sigma_{2,k_0}^{\text{out}} := \{(u_{1,2}, v_{1,2}, u_{k,2}, v_{k,2}, r_2) : u_{1,2} = \delta^{-1/3}\}. \tag{1.88}$$

The resulting geometry is illustrated in Figure 1.7. To define the transition map Π_2 in K_2, we consider initial conditions in a small neighbourhood R_2 around the point q_0.

Lemma 1.40. *The invariant set* $\{u_{k,2} = 0 = v_{k,2},\ 2 \le k \le k_0\}$ *is linearly stable under the flow of* (1.80) *if*

$$\frac{8^3 a^6}{\pi^6}\varepsilon_0 < \delta < \frac{\varepsilon_0}{\rho^3}. \tag{1.89}$$

Proof. Differentiation of (1.80c) with respect to $u_{k,2}$ shows that the linear stability condition we require is

$$\frac{b_k}{4A^2} + 2^{1/2} u_{1,2}(t) < 0 \quad \text{or, slightly stronger,} \quad u_{1,2}(t) < \frac{\pi^2}{8A^2} \tag{1.90}$$

for $2 \le k \le k_0$ in (1.80c), as b_k is negative and decreasing with k; recall (1.5). Given that $u_{1,2}(T_2) = \delta^{-1/3}$ in $\Sigma_{2,k_0}^{\text{out}}$, where T_2 denotes the (finite) transition time of the special orbit γ_2 between $\Sigma_{2,k_0}^{\text{in}}$ and $\Sigma_{2,k_0}^{\text{out}}$, γ_2 satisfies $u_{1,2}(t) \le \delta^{-1/3}$ throughout. Thus, it is sufficient to have

$$\delta^{-1/3} < \frac{\pi^2}{8a^2\varepsilon^{1/3}}, \tag{1.91}$$

where we have made use of $A = a\varepsilon^{1/6}$. We can simplify the last inequality to

$$\delta > \frac{8^3 a^6}{\pi^6}\varepsilon; \tag{1.92}$$

moreover, since $\varepsilon \in [0, \varepsilon_0)$, it is sufficient to have

$$\delta > \frac{8^3 a^6}{\pi^6} \varepsilon_0, \tag{1.93}$$

which places a lower bound on δ. Finally, the upper bound in the statement of the lemma follows from the definition of δ in chart K_1. □

Remark 1.41. The linear stability condition in (1.89) can be satisfied by restricting ε_0 on the left-hand-side of the condition so that δ can be chosen sufficiently small for the analysis in charts K_1 and K_3 to hold, and then choosing ρ small enough for the upper bound on the right-hand side to be satisfied.

Proposition 1.42. *The transition map $\Pi_{2,k_0} : \Sigma^{\text{in}}_{2,k_0} \to \Sigma^{\text{out}}_{2,k_0}$ is well-defined in a neighbourhood of the point q_0, see Figure 1.7, which maps diffeomorphically to a neighbourhood of $\Pi_{2,k_0}(q_0)$, where*

$$\Pi_{2,k_0}\left(q_0\right) = \left(\delta^{-1/3}, -\Omega_0 + 2^{1/2}\delta^{1/3} + \mathcal{O}(\delta), \mathbf{0}, \mathbf{0}, 0\right).$$

Moreover, $|u_{k,2}|$ and $|v_{k,2}|$ are non-increasing under Π_{2,k_0}.

Proof. Given Lemma 1.40, system (1.80) can be considered a regular perturbation of the Riccati equation in (1.81) in a sufficiently small neighbourhood around q_0. The above assertions then follow from Proposition 1.39 and regular perturbation theory. □

We can also derive estimates on the higher-order modes $\{u_{k,2}, v_{k,2}\}$ during the transition from $\Sigma^{\text{in}}_{2,k_0}$ to $\Sigma^{\text{out}}_{2,k_0}$. Consider a point $q_1 = \left(u_{1,2}(0), \delta^{-2/3}, u_{k,2}(0), v_{k,2}(0), r_2(0)\right)$, with $2 \le k \le k_0$, close to q_0. Since orbits of the full system, Equation (1.80), are regular perturbations of the orbit γ_2, the transition time $T_2(q_1)$ for the orbit initiated in q_1 will be equal to T_2, the transition time for γ_2, to leading order:

$$T_2(q_1) = T_2 + \mathcal{O}\left(u_{k,2}(0), v_{k,2}(0), r_2(0)\right). \tag{1.94}$$

The lower bound on δ in (1.89) then yields the following estimates on $u_{k,2}$ and $v_{k,2}$.

Lemma 1.43. *Let q_1 be as above. Then, the following estimates hold for $u_{k,2}(t)$ and $v_{k,2}(t)$ for any $t \in \left[0, T_2(q_1)\right]$:*

$$|u_{k,2}(t)| \le \exp\left(\frac{b_k}{16a^2 r_2(0)} t\right)|u_{k,2}(0)| + \frac{16a^2 r_2(0)}{|b_k|}\left[|v_{k,2}(0)| + \left(1 + \frac{4a^2 r_2(0)^2}{|b_k|}\right)\sigma\right] \quad \text{and} \tag{1.95}$$

$$|v_{k,2}(t)| \le \exp\left(\frac{b_k}{4a^2} r_2^2(0)\, t\right)|v_{k,2}(0)| + \frac{4a^2 r_2(0)^2 \sigma}{|b_k|} \tag{1.96}$$

for some constant $C > 0$ and $0 < \kappa \le \sigma < 1$, where κ is as in (1.30).

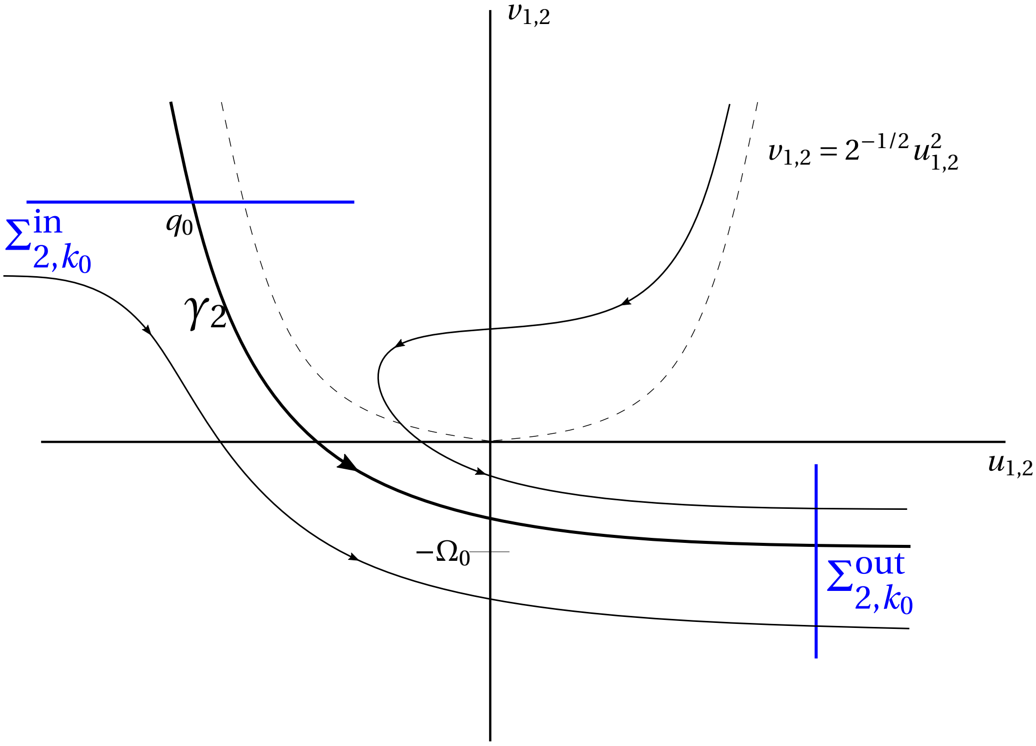


Figure 1.7: The dynamics in chart K_2 on the invariant plane $\{u_{k,2} = 0 = v_{k,2}\} \cap \{r_2 = 0\}$. For suitably chosen initial conditions and sufficiently small $r_2 = \varepsilon^{1/3}$, the general dynamics of (1.80) is a regular perturbation of that singular limit.

Proof. For $(\tilde{u}_{1,2}, \ldots, \tilde{u}_{k_0,2}, \tilde{v}_{1,2}, \ldots \tilde{v}_{k_0,2})$ in

$$\mathscr{B}_2 = \Big\{(\tilde{u}_{1,2}, \ldots, \tilde{u}_{k_0,2}, \tilde{v}_{1,2}, \ldots, \tilde{v}_{k_0,2}) : \tilde{u}_{k,2}, \tilde{v}_{k,2} \in C([0, T_2(q_1)]), \text{ with}$$
$$\sup_{[0,T_2]} |\tilde{u}_{k,2}(t)| \le C_k, \quad \sup_{[0,T_2(q_1)]} |\tilde{v}_{k,2}(t)| \le C_k \text{ for } 1 \le k \le k_0 \text{ and } \sum_{k=1}^{k_0} C_k^2 \le \tilde{\sigma}\Big\},$$

consider $M_k(\tilde{u}_{2,2}, \ldots \tilde{u}_{k_0,2}) = \sum_{i,j=2}^{k_0} \eta_{i,j}^k \tilde{u}_{j,2}\tilde{u}_{1,2}$ and the higher-order terms $H_{k,2}^u = H_{k,2}^u(\tilde{u}_{1,2}, \ldots, \tilde{u}_{k_0,2}, \tilde{v}_{1,2}, \ldots \tilde{v}_{k_0,2})$ and $H_{k,2}^v = H_{k,2}^v(\tilde{u}_{1,2}, \ldots, \tilde{u}_{k_0,2}, \tilde{v}_{1,2}, \ldots \tilde{v}_{k_0,2})$. Thus, we define a map $\mathscr{N}_2$ via $(\tilde{u}_{1,2}, \ldots, \tilde{u}_{k_0,2}, \tilde{v}_{1,2}, \ldots v_{k_0,2}) \mapsto (u_{1,2}, \ldots, u_{k_0,2}, v_{1,2}, \ldots v_{k_0,2})$, where $(u_{1,2}, \ldots, u_{k_0,2}, v_{1,2}, \ldots v_{k_0,2})$ are solutions of (1.80). To obtain the above estimates, we shall show that $\mathscr{N}_2 : \mathscr{B}_2 \to \mathscr{B}_2$. As $r_2(t)$ is constant in chart K_2, from (1.80d) we directly conclude

$$\begin{aligned}|v_{k,2}(t)| &\le \exp\left(\frac{b_k}{4A^2} r_2^3(0)\, t\right)|v_{k,2}(0)| + \exp\left(\frac{b_k}{4A^2} r_2^3(0)\, t\right)\int_0^t \exp\left(-\frac{b_k}{4A^2} r_2^3(0)\, s\right)|H_{k,2}^v|\, ds \\ &\le \exp\left(\frac{b_k}{4A^2} r_2^3(0)\, t\right)|v_{k,2}(0)| + \frac{4A^2 r_2(0)}{|b_k|}\sigma\end{aligned}$$

for all $t \in [0, T_2(q_1)]$, where $|H_{k,2}^v| \le C r_2(0)^4 \tilde{\sigma} \le r_2(0)^4 \sigma$. Applying the variation of constants formula to (1.80c), we find

$$|u_{k,2}(t)| \le \exp\left(\frac{(2-\sqrt{2})b_k}{8A^2}t\right)|u_{k,2}(0)| + \frac{8A^2}{(2-\sqrt{2})|b_k|}\left(\sup_{t\in[0,T_2(q_1)]}|v_{k,2}(t)| + C\tilde{\sigma}\right)$$
$$\le \exp\left(\frac{b_k}{16A^2}t\right)|u_{k,2}(0)| + \frac{16A^2}{|b_k|}\left[|v_{k,2}(0)| + \left(1+\frac{4a^2r_2(0)^2}{|b_k|}\right)\sigma\right]$$

for all $t \in [0, T_2(q_1)]$. Thus, for appropriately chosen $0 < r_2(0) < 1$ and $0 < \sigma < 1$, we obtain that $\mathcal{N}_2 : \mathcal{B}_2 \to \mathcal{B}_2$, as claimed, which implies (1.95) and (1.96). □

Remark 1.44. Since $A = a\varepsilon^{1/6} = a(r_2(0))^{1/2}$, the first term in (1.95) is equal to $\exp\left(-\frac{c}{r_2(0)}\right)|u_{k,2}|$ at $t = T_2(q_1)$, with $c > 0$ a constant, while the second term has the form of an $\mathcal{O}(r_2(0))$-correction.

In the corresponding result for $v_{k,2}(t)$ in (1.96), at $t = T_2(q_1)$ we have the bound $\exp\left(-cr_2^2(0)\right)|v_{k,2}(0)| \approx |v_{k,2}(0)|$ for small $r_2(0)$, as we are considering here.

Taking more general higher-order terms of the form $H^v = H^v(u^2, uv, v^2)$ in (1.2), we would find $H^v_{k,2}$ to be of the order $\mathcal{O}(r_2(0)^2)$; the second term in the estimate in (1.96) for $v_{k,2}(t)$ would read $4a^2\sigma/|b_k|$, which is uniformly bounded in $r_2(0)$ and k.

1.5.3 Chart K_3

In chart K_3, the blow-up transformation in (1.50) reads

$$u_1 = r_3, \quad v_1 = r_3^2 v_{1,3}, \quad u_k = r_3 u_{k,3}, \quad v_k = r_3^2 v_{k,3}, \quad \text{and} \quad \varepsilon = r_3^3\varepsilon_3.$$

After desingularising by dividing out a factor of r_3 from the resulting vector field, we obtain

$$r_3' = r_3F_3, \tag{1.97a}$$
$$v_{1,3}' = -2F_3v_{1,3} - 2^{1/2}\varepsilon_3, \tag{1.97b}$$
$$u_{k,3}' = \left(-F_3 + \frac{b_k}{4A^2}\varepsilon_3^{1/3} + 2^{1/2}\right)u_{k,3} - v_{k,3} + \sum_{i,j=2}^{k_0}\eta^k_{i,j}u_{i,3}u_{j,3} + H^u_{k,3}, \tag{1.97c}$$
$$v_{k,3}' = \left(-2F_3 + \frac{b_k}{4A^2}r_3^3\varepsilon_3^{4/3}\right)v_{k,3} + \varepsilon_3H^v_{k,3}, \tag{1.97d}$$
$$\varepsilon_3' = -3F_3\varepsilon_3, \tag{1.97e}$$

where

$$F_3 = F_3(r_3, v_{1,3}, u_{k,3}, v_{k,3}, \varepsilon_3) = -v_{1,3} + 2^{-1/2} + 2^{-1/2}\sum_{j=2}^{k_0}u_{j,3}^2 + H^u_{1,3},$$

with

$$H^u_{1,3} = \mathcal{O}\left(r_3\varepsilon_3, r_3^2v_{1,3}^2, r_3^2v_{j,3}^2, r_3v_{1,3}, r_3u_{j,3}v_{j,3}, r_3u_{j,3}^2, r_3u_{i,3}u_{j,3}u_{l,3}\right),$$
$$H^u_{k,3} = \mathcal{O}\left(r_3^2v_{1,3}v_{i,3}, r_3^2v_{i,3}v_{j,3}, r_3v_{i,3}, r_3u_{i,3}v_{1,3}, r_3u_{i,3}v_{j,3}, r_3u_{i,3}, r_3u_{i,3}u_{j,3}, r_3u_{i,3}u_{j,3}u_{l,3}\right), \quad \text{and}$$

$H^v_{k,3} = \mathcal{O}\left(r_3^4 v_{1,3} v_{i,3}, r_3^4 v_{i,3} v_{j,3}\right),$

for $2 \le i, j, l \le k_0$. As in K_1, we can rewrite (1.97) in the form

$$r_3' = r_3 F_3, \tag{1.98a}$$
$$v_{1,3}' = -2F_3 v_{1,3} - 2^{1/2}\varepsilon_3, \tag{1.98b}$$
$$u_{k,3}' = \left(-F_3 + \frac{b_k}{4A^2}\left(\varepsilon_3^{1/3}\right) + 2^{1/2}\right) u_{k,3} - v_{k,3} + \sum_{i,j=2}^{k_0} \eta^k_{i,j} u_{i,3} u_{j,3} + H^u_{k,3}, \tag{1.98c}$$
$$v_{k,3}' = \left(-2F_3 + \frac{b_k}{4A^2} r_3^3 \left(\varepsilon_3^{1/3}\right)^4\right) v_{k,3} + \varepsilon_3 H^v_{k,3}, \tag{1.98d}$$
$$\left(\varepsilon_3^{1/3}\right)' = -F_3\left(\varepsilon_3^{1/3}\right), \tag{1.98e}$$

for $2 \le k \le k_0$.

As already mentioned, the part $\gamma_3 := \kappa_{23}(\gamma_2)$ of the orbit γ_2 from chart K_2 with $u_{1,2} > 0$, transformed in K_3 has the expansion

$$\gamma_3 = \left(0, -\Omega_0 \varepsilon_3^{2/3} + 2^{1/2}\varepsilon_3 + \mathcal{O}\left(\varepsilon_3^{5/3}\right), \mathbf{0}, \mathbf{0}, \varepsilon_3\right)$$

as $\varepsilon_3 \to 0$. Thus, we see that γ_3 approaches the origin in chart K_3. It hence follows that the centre manifold $M_{k_0,1}$ from chart K_1 passes through a neighbourhood of the origin, which is a hyperbolic steady state for (1.98).

Let $w_k := (0, 0, \ldots, 1, \ldots, 0)$, with $2 \le k \le k_0$, denote the vector with $k_0 - 1$ components which are all equal to 0, except for the $(k-1)$-th which equals 1. Using this notation, a direct calculation shows the following result.

Lemma 1.45. *The origin is a hyperbolic steady state of Equation* (1.98)*, with the following eigenvalues and eigenvectors in the corresponding linearisation:*

- *the simple eigenvalue* $\frac{\sqrt{2}}{2}$ *with eigenvector* $(1, 0, \mathbf{0}, \mathbf{0}, 0)$*, corresponding to* r_3*;*
- *the simple eigenvalue* $-\sqrt{2}$ *with eigenvector* $(0, 1, \mathbf{0}, \mathbf{0}, 0)$*, corresponding to* $v_{1,3}$*;*
- *the eigenvalue* $\frac{\sqrt{2}}{2}$ *with multiplicity* $k_0 - 1$ *and eigenvectors* $(0, 0, w_k, \mathbf{0}, 0)$*, corresponding to* $u_{k,3}$*,* $2 \le k \le k_0$*;*
- *the eigenvalue* $-\sqrt{2}$ *with multiplicity* $k_0 - 1$ *and eigenvectors* $(0, 0, \frac{\sqrt{2}}{3} w_k, w_k, 0)$*, corresponding to* $v_{k,3}$*,* $2 \le k \le k_0$*; and*
- *the simple eigenvalue* $\frac{-\sqrt{2}}{2}$ *with eigenvector* $(0, 0, \mathbf{0}, \mathbf{0}, 1)$*, corresponding to* $\varepsilon_3^{1/3}$*.*

Remark 1.46. Since

$$-\frac{\sqrt{2}}{2} = -\sqrt{2} + \frac{\sqrt{2}}{2},$$

the eigenvalues of (1.98) are in resonance. Potential second-order resonant terms are $r_3 v_{1,3}$, $r_3 v_{k,3}$, and $u_{i,3} v_{j,3}$. While resonances are also observed in the singularly perturbed planar fold [34], the resonant terms differ, which is due to us formulating the governing equations in chart K_3 in terms of $\varepsilon_3^{1/3}$. Furthermore, the higher dimensionality of (1.98), as compared to the equivalent system in their chart K_3, allows for a richer resonance structure which will affect the higher order asymptotics of $\mathscr{C}_\varepsilon$. This however outside the scope of this work.

The entry section $\Sigma^{\text{in}}_{3,k_0}$ in chart K_3, which is obtained by transformation of the exit section $\Sigma^{\text{out}}_{2,k_0}$ from K_2, is given by

$$\Sigma^{\text{in}}_{3,k_0} = \{(r_3, v_{1,3}, u_{k,3}, v_{k,3}, \mu_3) : \varepsilon_3 = \delta\}, \tag{1.99}$$

where we consider the set of initial conditions

$$R_3 = \{(r_3, v_{1,3}, u_{k,3}, v_{k,3}, \varepsilon_3) \mid r_3 \in [0,\rho], v_{1,3} \in [-\beta, \beta], \\ |u_{k,3}| \le C_{u_{k,3}}, |v_{k,3}| \le C_{v_{k,3}}, \text{ and } \varepsilon_3 = \delta\} \subset \Sigma^{\text{in}}_{3,k_0}. \tag{1.100}$$

Here, $\beta, C^{\text{in}}_{u_{k,3}}$, and $C^{\text{in}}_{v_{k,3}}$, for $2 \le k \le k_0$, are appropriately defined small constants. We also define the exit chart

$$\Sigma^{\text{out}}_{3,k_0} := \left\{(r_3, v_{1,3}, u_{k,3}, v_{k,3}, \varepsilon_3) : r_3 = \rho\right\}. \tag{1.101}$$

Our aim is to describe the transition map $\Pi_3 : R_3 \to \Sigma^{\text{out}}_{3,k_0}$. Therefore, since F_3 is bounded away from zero near the origin, we can divide the vector field in (1.98) by F_3, which results in

$$r_3' = r_3, \tag{1.102a}$$

$$v_1' = -2v_1 - 2^{1/2}\frac{\left(\varepsilon_3^{1/3}\right)^3}{F_3}, \tag{1.102b}$$

$$u_k' = \left(-1 + \frac{b_k}{4A^2}\frac{\varepsilon_3^{1/3}}{F_3} + \frac{2^{1/2}}{F_3}\right)u_k - \frac{1}{F_3}v_k + \frac{1}{F_3}\sum_{i,j=2}^{k_0}\eta^k_{i,j}u_i u_j + \frac{1}{F_3}H^u_k, \tag{1.102c}$$

$$v_k' = \left(-2 + \frac{b_k}{4A^2}\frac{r^3\left(\varepsilon_3^{1/3}\right)^4}{F_3}\right)v_k + \frac{\left(\varepsilon_3^{1/3}\right)^3}{F_3}H^v_k, \tag{1.102d}$$

$$\left(\varepsilon_3^{1/3}\right)' = -\varepsilon_3^{1/3}, \tag{1.102e}$$

where the prime denotes differentiation with respect to the new, rescaled, time variable. (We have suppressed the subscript 3 in (1.102) for convenience of notation, and will do so for the remainder of the section.)

The above rescaling of time by F_3 results in the eigenvalues of the linearisation about the origin being rescaled by a factor $2^{-1/2}$.Lemma 1.45 hence now implies the following:

Lemma 1.47. *The origin is a hyperbolic steady state of* (1.102)*, with the following eigenvalues in the corresponding linearisation:*

- *the simple eigenvalue* 1*, corresponding to* r_3*;*
- *the simple eigenvalue* -2*, corresponding to* v_1*;*
- *the eigenvalue* 1 *with multiplicity* $k_0 - 1$*, corresponding to* u_k*,* $2 \le k \le k_0$*;*
- *the eigenvalue* -2 *with multiplicity* $k_0 - 1$*, corresponding to* v_k*,* $2 \le k \le k_0$*; and*

- *the simple eigenvalue* -1*, corresponding to* $\varepsilon_3^{1/3}$.

The associated eigenvectors are as given in Lemma 1.45.

To obtain estimates for the transition map Π_3, we follow a procedure that is analogous to that in [34] for this chart. We begin by separating out terms containing r_3 in (1.102). To that end, we expand

$$\frac{1}{F_3(v_1, u_k, r_3)} = G_3(v_1, u_k) + \mathcal{O}(r_3), \tag{1.103}$$

where

$$G_3(v_1, u_k) = \frac{1}{2^{-1/2} - v_1 + 2^{-1/2}\sum_{j=2}^{k_0} u_j^2}. \tag{1.104}$$

With the above notation, we can rewrite system (1.102) in a form that separates the terms with r_3.

Lemma 1.48. *For* $r_3 \geq 0$ *sufficiently small,* (1.102) *can be written as*

$$r_3' = r_3, \tag{1.105a}$$
$$v_1' = -2v_1 - 2^{1/2}\varepsilon_3 G_3 + \varepsilon_3 r_3 J_{v_1}, \tag{1.105b}$$
$$u_k' = \left(-1 + \frac{b_k}{4A^2}(\varepsilon_3^{1/3})G_3 + 2^{1/2}G_3\right)u_k - G_3 v_k + G_3\sum_{i,j=2}^{k_0}\eta_{i,j}^k u_i u_j + r_3 J_{u_k}, \tag{1.105c}$$
$$v_k' = \left(-2 + \frac{b_k}{4A^2}(\varepsilon_3^{1/3})^4 r_3 G_3\right)v_k + \varepsilon_3 r_3 J_{v_k}, \tag{1.105d}$$
$$(\varepsilon_3^{1/3})' = -(\varepsilon_3^{1/3}), \tag{1.105e}$$

where J_{v_1}, J_{u_k}, *and* J_{v_k} *are smooth functions.*

We now have the following result on the transition map Π_3.

Proposition 1.49. *The transition map* $\Pi_3 : R_3 \to \Sigma_{3,k_0}^{\text{out}}$ *is well-defined. Let* $(r_3^{\text{in}}, v_1^{\text{in}}, u_k^{\text{in}}, v_k^{\text{in}}, \delta) \in R_3$*, as defined in* (1.100)*, where* $k = 2, \ldots, k_0$*, and let* T_3 *be the corresponding transition time between* $\Sigma_{3,k_0}^{\text{in}}$ *and* $\Sigma_{3,k_0}^{\text{out}}$ *under the flow of* (1.105)*. Then, the map* Π_3 *is given by*

$$\Pi_3(r_3^{\text{in}}, v_1^{\text{in}}, u_k^{\text{in}}, v_k^{\text{in}}, \delta) = \left(\rho, \Pi_{3,v_1}^{k_0}, \Pi_{3,u_k}^{k_0}, \Pi_{3,v_k}^{k_0}, \delta^{1/3}\frac{r_3^{\text{in}}}{\rho}\right),$$

where

$$\left|\Pi_{3,v_1}(r_3^{\text{in}}, v_1^{\text{in}}, u_k^{\text{in}}, v_k^{\text{in}}, \delta)\right| \leq \left(\frac{r_3^{\text{in}}}{\rho}\right)^2\left[|v_1^{\text{in}}| + C_{v_{1,3}}^{\text{out}}(1 + r_3^{\text{in}}\log r_3^{\text{in}})\right], \tag{1.106}$$
$$\left|\Pi_{3,u_k}(r_3^{\text{in}}, v_1^{\text{in}}, u_k^{\text{in}}, v_k^{\text{in}}, \delta)\right| \leq C_{u_{k,3}}^{\text{out}}, \quad \textit{and} \tag{1.107}$$
$$\left|\Pi_{3,v_k}(r_3^{\text{in}}, v_1^{\text{in}}, u_k^{\text{in}}, v_k^{\text{in}}, \delta)\right| \leq C_{v_{k,3}}^{\text{out}}, \tag{1.108}$$

for positive constants $C_{v_{1,3}}^{\text{out}}$, $C_{u_{k,3}}^{\text{out}}$, *and* $C_{v_{k,3}}^{\text{out}}$.

Proof. From (1.105), we have that

$$r_3(t) = r_3^{\rm in} e^t \quad \text{and} \quad \varepsilon_3(t) = \delta e^{-3t}, \tag{1.109}$$

which gives the transition time

$$T_3 = \log\left(\frac{\rho}{r_3^{\rm in}}\right) \tag{1.110}$$

between $\Sigma^{\rm in}_{3,k_0}$ and $\Sigma^{\rm out}_{3,k_0}$.

For $\sum_{k=2}^{k_0} |\tilde{u}_k(t)|^2 \le \sigma$ and $\sum_{k=2}^{k_0} |\tilde{v}_k(t)|^2 \le \sigma$, with $0<\sigma\le 1$, and $|v_1(t)| \le 1/(2\sqrt{2})$ for $t \in [0, T_3]$, consider $J_{v_1} = J_{v_1}(v_1, \tilde{u}_k, \tilde{v}_k)$ and $J_{l_k} = J_{l_k}(v_1, \tilde{u}_k, \tilde{v}_k)$, with $l = u, v$ and $k = 2, \dots, k_0$, as well as $G_3 = G_3(v_1, \tilde{u}_k)$. We observe that

$$\begin{aligned} G_3(v_1(t), \tilde{u}_k(t)) &\le \frac{1}{2^{-1/2} - |v_1(t)|} = 2^{1/2} + 2^{1/2} \sum_{n=1}^{\infty} 2^{\frac{n}{2}} |v_1(t)|^n \\ &\le 2^{1/2}\left(1 + 2^{1/2} \frac{|v_1(t)|}{1-\sqrt{2}|v_1(t)|}\right) \le 2^{1/2}\big(1 + 2^{3/2}|v_1(t)|\big) \end{aligned} \tag{1.111}$$

for $|v_1(t)| \le 1/(2\sqrt{2})$, and

$$G_3(v_1(t), \tilde{u}_k(t)) \ge \frac{1}{2^{-1/2} + |v_1(t)| + 2^{-1/2}\sigma} =: C_1 \ge 1. \tag{1.112}$$

Then, using the boundedness of G_3 and J_{v_k}, from Equation (1.105d) for v_k we obtain directly

$$|v_k(t)| \le e^{-2t+b_3(t)}|v_k(0)| + C\int_0^t e^{-2(t-s)+B_3(t)-B_3(s)} \varepsilon_3(s) r_3(s)\, ds \le e^{-2t}|v_k(0)| + \frac{C\rho a^2 \sigma}{|b_k|},$$

where $B_3(t) = r_3^{\rm in} \delta^{4/3} b_k (1 - e^{-3t})/(12A^2)$. To determine the asymptotic behaviour of v_1, we define a new variable z by

$$v_1 = e^{-2t}\left(v_1^{\rm in} + z\right); \tag{1.113}$$

in particular, for $t = 0$, it follows that $z(0) = 0$. A direct calculation yields

$$\begin{aligned} v_1' &= e^{-2t} z' - 2e^{-2t}\left(v_1^{\rm in} + z\right), \\ -2v_1 - \delta e^{-3t} G_3 + r_3^{\rm in} \delta e^{-2t} J_{v_1} &= e^{-2t} z' - 2e^{-2t}\left(v_1^{\rm in} + z\right), \quad \text{and} \\ -2e^{-2t}\left(v_1^{\rm in} + z\right) + \delta e^{-3t} G_3 + r_3^{\rm in} \delta e^{-2t} J_{v_1} &= e^{-2t} z' - 2e^{-2t}\left(v_1^{\rm in} + z\right) \end{aligned}$$

or, equivalently,

$$z' = -2^{1/2} e^{-t} \delta G_3 + r_3^{\rm in} \delta J_{v_1}. \tag{1.114}$$

Then, the boundedness of G_3 and J_{v_1} implies

$$|z(t)| \le C\delta(1 - e^{-t} + r_3^{\rm in} t).$$

Reverting to the original variable v_1 via (1.113), we find

$$|v_1(t)| \le e^{-2t}\left[|v_1^{\text{in}}| + C\delta(1-e^{-t}+r_3^{\text{in}}t)\right] \le |v_1^{\text{in}}| + 2\delta C_2 \le 1/(2\sqrt{2}) \quad \text{for all } t\in[0,T_3]$$

with sufficiently small $|v_1^{\text{in}}|$, and

$$|v_1(T_3)| \le \left(\frac{r_3^{\text{in}}}{\rho}\right)^2 \left[|v_1^{\text{in}}| + C\delta\left(1 + r_3^{\text{in}}\log\left(\frac{\rho}{r_3^{\text{in}}}\right)\right)\right], \tag{1.115}$$

which proves the first point of the theorem.

Next, we show that u_k remains bounded throughout the transition through chart K_3. Once again, we perform an estimate using the variation of constants formula. We have

$$\begin{aligned} u_k(t) &= \exp\left(\int_0^t U_k(\tau)d\tau\right)u_k^{\text{in}} \\ &+ \int_0^t \exp\left(\int_s^t U_k(\tau)d\tau\right)\left(-G_3(v_1,\tilde{u}_k)v_k + G_3(v_1,\tilde{u}_k)\sum_{i,j=2}^{k_0}\eta_{i,j}^k\tilde{u}_i\tilde{u}_j + r_3 J_{u_k}\right)ds, \end{aligned}$$

where

$$U_k(\tau) = -1 + \frac{b_k}{4A^2}\delta^{1/3}e^{-\tau}G_3(v_1(\tau),\tilde{u}_k(\tau)) + 2^{1/2}G_3(v_1(\tau),\tilde{u}_k(\tau)).$$

Using our assumptions on $\tilde{v}_k$ and $\tilde{u}_k$, the estimate for $G_3(v_1,\tilde{u}_k)$ in (1.111) and the fact that $b_k<0$, we find

$$\begin{aligned} \mathscr{I}_1(s) &:= \int_s^t U_k(\tau)d\tau \\ &= s - t + \frac{b_k}{4A^2}\delta^{1/3}\int_s^t e^{-\tau}G_3(v_1(\tau),\tilde{u}_k(\tau))d\tau + 2^{1/2}\int_s^t G_3(v_1(\tau),\tilde{u}_k(\tau))d\tau \\ &\le t - s + \frac{b_k}{4A^2}\delta^{1/3}C_1\left(e^{-s}-e^{-t}\right) + 4\int_s^t |v_1(\tau)|d\tau \end{aligned} \tag{1.116}$$

for $0\le s\le t\le T_3$. To estimate the integral, we observe that

$$|v_1(\tau)| \le e^{-2\tau}\left(|v_1^{\text{in}}| + C_2\delta(1+\tau)\right)$$

and write

$$\int_s^t |v_1(\tau)|d\tau \le \frac{1}{4}(2|v_1^{\text{in}}|+3C_2\delta)(e^{-2s}-e^{-2t}) + C_2\frac{\delta}{2}(e^{-2s}s - e^{-2t}t)$$

The inequality in (1.116) becomes

$$\mathscr{I}_1(s) \le t - s + \frac{b_k}{4A^2}\delta^{1/3}C_1\left(e^{-s}-e^{-t}\right) + (2|v_1^{\text{in}}|+3C_2\delta)(e^{-2s}-e^{-2t}) + 2C_2\delta(e^{-2s}s - e^{-2t}t).$$

Thus,

$$\exp(\mathscr{I}_1(0)) \le C \exp\Big(t + \frac{b_k}{4A^2}\delta^{1/3}C_1\big(1-e^{-t}\big)\Big),$$

where $C = \exp\big(2|v_1^{\rm in}| + 5C_2\delta\big)$. For the second term in $u_k(t)$, using that $t \le e^t$ and $1 \le e^t$ for $t \ge 0$, we have

$$\begin{aligned}\int_0^t \exp(\mathscr{I}_1(s))\,ds &\le \mathscr{I}_2(t)\int_0^t \exp\Big(-s + \frac{b_k}{4A^2}\delta^{1/3}C_1e^{-s} + C_3e^{-2s} + 2C_2\delta e^{-2s}s\Big)ds \\ &\le \mathscr{I}_2(t)\int_0^t e^{-s}\exp\Big(\Big[\frac{b_k}{4A^2}\delta^{1/3}C_1 + C_4\delta\Big]e^{-s}\Big)ds \\ &= \mathscr{I}_2(t)\frac{4A^2}{C_1|b_k|\delta^{1/3} - 4A^2C_4}\Big[\exp\Big(\Big[\frac{b_k\delta^{1/3}}{4A^2}C_1 + C_4\delta\Big]e^{-t}\Big) - \exp\Big(\frac{b_k\delta^{1/3}}{4A^2}C_1 + C_4\Big)\Big] \\ &\le C_5\frac{4A^2}{C_1|b_k|\delta^{1/3} - 4A^2C_4}e^t \le C_6\frac{a^2}{|b_k|}\varepsilon^{1/3}\frac{\rho}{r_3^{\rm in}} \le \frac{C_7\rho a^2}{|b_k|},\end{aligned}$$

where $C_3 = 2|v_1^{\rm in}| + 3C_2\delta$ and

$$\mathscr{I}_2(t) = \exp\Big(t - \frac{b_k}{4A^2}\delta^{1/3}C_1e^{-t} - C_3e^{-2t} - 2C_2\delta e^{-2t}t\Big).$$

Our estimates for v_1 and v_k then yield

$$|u_k(t)| \le C_1|u_k(0)| + \frac{C_2\rho a^2}{|b_k|}\Big(|v_k(0)| + \frac{C\rho a^2\sigma}{|b_k|} + \sigma(1+\rho)\Big).$$

For sufficiently small $|u_k(0)|$, $|v_k(0)|$, and ρ, we obtain

$$\sum_{k=2}^{k_0}|u_k(t)|^2 \le \sigma \quad \text{and} \quad \sum_{k=2}^{k_0}|v_k(t)|^2 \le \sigma \quad \text{for all } t \in [0, T_3].$$

Thus, applying a fixed point argument as is the estimates in chart K_1 we obtain the required estimates. □

1.5.4 Proof of main result

Let us now combine the analysis in the three charts K_1, K_2, and K_3 to give the proof of Theorem 1.11. For $2 \le k \le k_0$, the initial conditions $(u_{1,1}^{\rm in}, r_1^{\rm in}, u_{k,1}^{\rm in}, v_{k,1}^{\rm in}, \varepsilon_1^{\rm in})$ in K_1 are assumed to lie in $R_1 \subset \Sigma_{1,k_0}^{\rm in}$, where R_1 is defined as in (1.76). If we apply the transition map Π_{1,k_0}, see Proposition 1.38, we find

$$\Pi_1(R_1) = \Big\{|u_{1,1}^{\rm out} + 2^{1/4}| \le C_{u_{1,1}}^{\rm out},\ r_1^{\rm out} \in [0,\rho],\ \varepsilon_1^{\rm out} = \delta, |u_{k,1}^{\rm out}| \le C_{k,1}^{\rm out}, \text{ and } |v_{k,1}^{\rm out}| \le C_{v_{k,1}}^{\rm out}\delta^{2/3}\Big\}.$$

Transformation of the above set to chart K_2 yields

$$\begin{aligned}\kappa_{12}\circ\Pi_1(R_1) = \Big\{u_{1,2}^{\rm in} = \delta^{-1/3}u_{1,1}^{\rm out}, v_{1,2}^{\rm in} = \delta^{-2/3},\ |u_{k,2}^{\rm in}| \le \delta^{-1/3}|u_{k,1}^{\rm out}|,\\ |v_{k,2}^{\rm in}| \le \delta^{-2/3}|v_{k,1}^{\rm out}|, \text{ and } r_2^{\rm in} \in [0, \delta^{1/3}\rho]\Big\},\end{aligned}$$

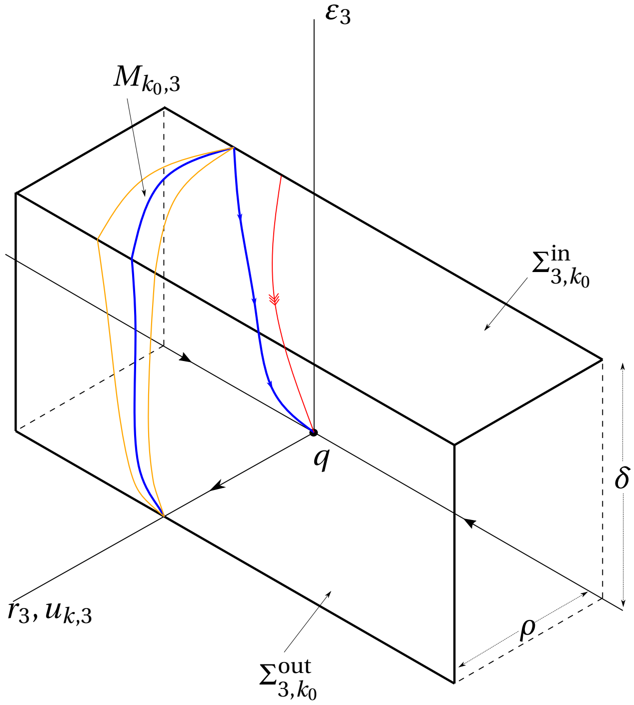


Figure 1.8: Dynamics in chart K_3. As the special orbit γ_2 from chart K_2, after transformation to K_2 via $\gamma_3 := \kappa_{23}(\gamma_2)$ (in red), passes through the origin q, the invariant manifold $M_{k_0,3}$ contains q. The transition map Π_3 is defined in a neighbourhood of the intersection $M_{k_0,3} \cap \Sigma^{\text{in}}_{3,k_0}$.

with $u_{k,1}$ and $v_{k,1}$ as above. Since the higher-order modes $\{u_{k,2}, v_{k,2}\}$ in K_2 do not grow, we may write

$$\Pi_2 \circ \kappa_{12} \circ \Pi_{1,k_0}(R_1) = \Big\{ u^{\text{out}}_{1,2} = \delta^{-1/3}, |v^{\text{out}}_{1,2} + c_1| \le C^{\text{out}}_{v_{1,2}}, \\ |u^{\text{out}}_{k,2}| \le |u^{\text{in}}_{k,2}|, |v^{\text{out}}_{k,2}| \le |v^{\text{in}}_{k,2}|, r^{\text{out}}_2 \in \big[0, \delta^{1/3}\rho\big] \Big\}. \quad (1.117)$$

Application of the change of coordinates κ_{23} yields

$$\kappa_{23} \circ \Pi_2 \circ \kappa_{12} \circ \Pi_1(R) = \\ \Big\{ r^{\text{in}}_3 \in [0,\varepsilon],\ v^{\text{in}}_{1,3} \in [-\beta,\beta],\ \varepsilon^{\text{in}}_3 = \delta, |u^{\text{in}}_{k,3}| = \delta^{1/3}|u^{\text{out}}_{k,2}| \le C^{\text{in}}_{u_{k,3}}, |v^{\text{in}}_{k,3}| = \delta^{2/3}|v^{\text{out}}_{k,2}| \le C^{\text{in}}_{v_{k,1}} \Big\},$$

where $\beta > 0$ is a small constant. Finally, we apply the map Π_3, see Proposition 1.49, to obtain

$$\Pi_3 \circ \kappa_{23} \circ \Pi_2 \circ \kappa_{12} \circ \Pi_1(R_1) = \\ \Big\{ r^{\text{out}}_3 = \rho,\ v^{\text{out}}_{1,3},\ \varepsilon^{\text{out}}_3 \in [0,\delta], |u^{\text{out}}_{k,3}| \le C^{\text{out}}_{u_{k,3}}, |v^{\text{out}}_{k,3}| \le C^{\text{out}}_{v_{k,3}} \Big\},$$

where, again, $v^{\text{out}}_{1,3}$ is as in Proposition 1.49. The result then follows, since the sections $\Sigma^{\text{in}}_{1,k_0}$ and $\Sigma^{\text{out}}_{3,k_0}$ are, in fact, equivalent to Δ^{in} and Δ^{out}, respectively, transformed into the coordinates of charts K_1 and K_3, respectively, and since the systems in (1.28) and (1.53) are equivalent for $\varepsilon > 0$ sufficiently small.

1.6 Conclusion and outlook

In this work, we have studied a fast-slow system of partial differential equations (PDEs) of reaction-diffusion type, Equation (1.1), under the assumption that a fold singularity is present at the origin in the fast kinetics. We have extended a family $S_{\varepsilon,\zeta}$ of slow manifolds for the system past that singularity by first performing a Galerkin discretisation and then applying the desingularisation technique known as blow-up [17] to the resulting system of ordinary differential equations (ODEs). As we have seen, our main result, Theorem 1.11, is analogous to what one would expect in the planar (finite-dimensional) setting [34]. While we have shown that the resulting Galerkin manifolds $\mathscr{C}_{\varepsilon,k_0}$ approximate the family $S_{\varepsilon,\zeta}$ away from the fold singularity, the crucial question of whether one can pass to the limit of $k_0 \to \infty$ and state that correspondence to the extension of $\mathscr{C}_{\varepsilon,k_0}$ via blow-up still remains open.

Correspondingly, the presence of an additional $2k_0 - 2$ equations, with k_0 arbitrarily large, after discretisation causes several challenges. In particular, the scaling of the spatial domain with (a fractional power of) ε is essential to our approach, and is required to obtain both well-defined and non-trivial dynamics in the singular limit after blow-up. That rescaling seems to be natural, since it can be recovered directly from the original system of PDEs in (1.1).

Specifically, taking $u = \varepsilon^{1/3}U$, $v = \varepsilon^{2/3}V$, $t = \varepsilon^{-1/3}\tau$, and $x = \varepsilon^{-1/6}X$, which is consistent with our scaling in (1.56), we obtain

$$\begin{aligned}
\partial_\tau U &= \partial_X^2 U - V + U^2 + \varepsilon^p H_u(U,V) && \text{on } (-a\varepsilon^{1/6}, a\varepsilon^{1/6}),\\
\partial_\tau V &= \varepsilon\partial_X^2 V - 1 + \varepsilon^q H_v(U,V) && \text{on } (-a\varepsilon^{1/6}, a\varepsilon^{1/6}),
\end{aligned} \tag{1.118}$$

for some $p, q > 0$. Equation (1.118) defines a system of PDEs on a domain shrinking to the origin as $\varepsilon \to 0$, as is to be expected due to the singular nature of (1.1). Denoting by $(U_\varepsilon, V_\varepsilon)$ solutions of (1.118), and using the boundedness of higher-order terms and the non-positivity of U_ε or the boundedness of U_ε^2, which can be achieved by considering a cut-off function, we obtain the following estimates:

$$\begin{aligned}
\|V_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \varepsilon\|\partial_X V_\varepsilon\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} &\le C\big(\|V_\varepsilon(0)\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{1/6}\big),\\
\|U_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \|\partial_X U_\varepsilon\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} &\le C\big(\|U_\varepsilon(0)\|^2_{L^2(\Omega_\varepsilon)} + \|V_\varepsilon(0)\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{1/6}\big),
\end{aligned}$$

where $\Omega_\varepsilon = (-a\varepsilon^{1/6}, a\varepsilon^{1/6})$ and C is some positive constant independent of ε. These estimates then imply that $U_\varepsilon(\cdot, \varepsilon^{1/6}\cdot) \rightharpoonup U_0$ in $L^2(0,T;H^1(-a,a))$, which is independent of X, and $V_\varepsilon(\cdot,\varepsilon^{1/6}\cdot) \rightharpoonup V_0$, $\varepsilon^{1/2}\partial_X V_\varepsilon(\cdot,\varepsilon^{1/6}\cdot) \rightharpoonup W$ in $L^2((0,T)\times(-a,a))$ as $\varepsilon \to 0$, for some $W \in L^2((0,T)\times(-a,a))$. Thus, in the limit as $\varepsilon \to 0$, we see that (U_0, V_0) satisfies the system of ODEs

$$\begin{aligned}
\frac{\mathrm{d}U}{\mathrm{d}\tau} &= -V + U^2,\\
\frac{\mathrm{d}V}{\mathrm{d}\tau} &= -1.
\end{aligned} \tag{1.119}$$

Equation (1.119) is precisely the Riccati equation which lies at the heart of the dynamics in our rescaling chart K_2.

A consequence of our rescaling of the spatial domain is, however, that the original fast-slow structure which is present in the discretised system, Equation (1.28), does not translate to our blow-up analysis in the three coordinate charts; in particular, there is no longer a direct correspondence between singular objects in those charts and the layer and reduced problems pre-blow-up.

As elaborated in Section 1.4.3, an additional challenge arises due to the finite-time blowup which can occur in (1.28) and which is due to the presence of additional slow variables v_k, $2 \leq k \leq k_0$ after Galerkin discretisation. To avoid that blowup, we defined an ε-dependent set of initial values $R^{\text{in}}(\varepsilon) \subset \Delta^{\text{in}}$, which we combined with careful estimates for the higher-order modes $\{u_k, v_k\}$ resulting from the discretisation. We conjecture that this blowup is, in essence, caused by additional fold singularities that can be reached before the principal singularity at the origin which has been our focus here. In particular, a future direction of research would be the desingularisation of larger submanifolds where normal hyperbolicity is lost in the Galerkin discretisation; for example, one could blow up the blue curve in Figure 1.2 or the surface in Figure 1.3 in the cases where $k_0 = 2$ or $k_0 = 3$, respectively. More generally, it would be interesting to investigate the relation between the various stability regions of critical manifolds after discretisation and the underlying systems of PDE.

Finally, we briefly place our work into the broader context of singular perturbation problems arising in an infinite-dimensional context. First, for fast-slow reaction-diffusion systems of the form in (5), we have recently gained a better understanding of transcritical points and generic fold points, including the results presented in this paper. In finite dimensions, such non-hyperbolic points are known to generate only a dichotomy of either fast jumps of trajectories or an exchange of stability between slow manifolds. Yet, it is known that more degenerate fold points, such as folded nodes or folded saddle-nodes, may generate extremely complicated local dynamics including oscillatory patterns even in fast-slow systems of ODEs. That classification is likely to become even more complex in the infinite-dimensional setting of (systems of) PDEs. Second, systems of the form in (5) represent one class of interesting PDEs, where small perturbation parameters and singular limits occur. Other classes involve fast reaction terms, small diffusion problems, or heterogeneous media with highly oscillatory coefficients, which all commonly appear in the context of reaction-diffusion systems. Once one goes beyond reaction-diffusion systems, there are vast classes of PDE-type singular perturbation problems arising across the sciences. From a mathematical viewpoint, it is immediately clear that, in any parametrised PDE model, one anticipates possible distinctions between normally hyperbolic behaviour, where a locally good approximation is achieved by linearisation, and a loss of normal hyperbolicity along submanifolds in parameter space. Therefore, there is a crucial need for developing techniques to tackle a loss of normal hyperbolicity in (systems of) PDEs. Our work is but one building block towards that more general effort. Last, but not least, we have not yet related our theoretical approach via Galerkin discretisation with the performance of various numerical methods for PDEs – we conjecture that there is a link between (a loss of) performance and the presence of singularities, or non-hyperbolic points, in nonlinear systems of PDEs.

Chapter 2

Geometric analysis for slow passage through Hopf bifurcations for PDEs

2.1 Introduction

To further explore the applicability of the approach described in chapter 1, in this chapter, we attempt a similar analysis with PDE version of the slow passage through dynamic Hopf bifurcation. Essentially, we study the normal form of a Hopf bifurcation with slowly varying bifurcation parameter with added diffusion, resulting in the PDE system

$$z_t = z_{xx} + (\mu + i)z - z|z|^2 + \varepsilon \hat{h}_0, \tag{2.1a}$$

$$\mu_t = \varepsilon(\mu_{xx} + 1), \tag{2.1b}$$

where $z(t,x) \in \mathbb{C}, \mu(t,x) \in \mathbb{R}, x \in [-a,a], \hat{h}_0 \in \mathbb{C}, \hat{h}_0 \neq 0$, accompanied by homogeneous Neumann boundary conditions. The term $\varepsilon \hat{h}_0$, for is introduced to break the $z \mapsto -z$ symmetry of the problem.

The analysis of the ODE version of the problem, as well as a multitude of references concerning research on it can be found in [25]. Our analysis mirrors that work, in that we first focus on the simplified Shishkova equation which captures many of the essential features of the problem, before proceeding with the full system (2.1). The ODE system in [25] is replaced by an arbitrarily large ODE system of $2k_0$ equations resulting from the truncation of the Galerkin expansion. A similar PDE-ODE system has been studied in [33]. The novelty of our approach in this chapter is that we our method can be applied in cases where both fast and slow components are PDEs; this is the reason μ is taken to be space dependent. This also implies that, in general, if starting with $\mu(x,0) < 0$ for $x \in [-a,a]$, then for each x_0, $\mu(x,t)$ will become positive at time $t_0 = t_0(x_0)$.

Our main result analogous and in broad terms, that will be made precise later, can be stated as follows:

Theorem 2.1. *Fix $k_0 \in \mathbb{N}$ and consider the truncated at k_0 Galerkin system of ODEs of* (2.1)*. Let $d(\varepsilon)$ be the distance between the stable and unstable slow manifolds of* *the truncated system. Then,*

$$d(\varepsilon) = \left(C\varepsilon^{1/2} + \mathcal{O}(\varepsilon)\right) e^{-\frac{1}{2\varepsilon}}$$

for a constant $C > 0$ *as* $\varepsilon \to 0$.

With regards to the question on whether a similar result can be stated for the untruncated system, the discussion in the introduction of the thesis and in the previous chapter on the double limit $k_0 \to \infty, \varepsilon \to 0$ is valid for this problem too.

2.2 Shishkova system

First we consider the simplified Shishkova system

$$\begin{aligned} z_t &= z_{xx} + (\mu + i)z + \varepsilon \hat{g}(\mu) && \text{in } (-a, a), && (2.2a)\\ \mu_t &= \varepsilon\left(\mu_{xx} + 1\right) && \text{in } (-a, a), && (2.2b)\\ z_x &= 0, \quad \mu_x = 0 && \text{at } x = \pm a, && (2.2c) \end{aligned}$$

where $z(x, t) \in \mathbb{C}$ and $\hat{g}(\mu)$ is an analytic function. The reaction-diffusion systems (2.2), (2.1) are locally well-posed in time by standard results [26, 4], generate a semi-flow on $L^2([-a, a]) \times L^2([-a, a])$ and enjoy favourable regularity properties. Numerical simulations of the PDE can be seen in Figure 2.1 and verify what we would expect: starting with initial condition $\mu(x) \approx -c, x \in [-a, a]$, where $c > 0$ is a constant and perturbing $\mu(x)$ so that it is not constant in space, μ increases and becomes positive at different times depending on $x \in [-a, a]$ and the instability in the fast component z_t does not set in until about when $\mu(x) \approx 1$ for all $x \in [-a, a]$.

Before we proceed, we introduce $s := \mu + i$ (Figure 2.4) in order eliminate the rotation in the fast component as $\mu(x, t)$ crosses from negative to positive values for $x \in [-a, a]$.

Remark 2.2. This translation is essential, as it shifts the imaginary eigenvalues of the steady state at the origin to 0, making it amenable to geometric desingularisation.

We write

$$\begin{aligned} z_t &= z_{xx} + sz + \varepsilon \hat{h}(s), && (2.3a)\\ s_t &= \varepsilon\left(s_{xx} + 1\right) && (2.3b) \end{aligned}$$

where $\hat{h}(s) = \hat{g}(s - i)$.

2.2.1 Galerkin descritisation

Our main result is a statement on the Galerkin discretisation of (2.2), concerning the distance of the stable and unstable manifolds at $s = 0$ (equivalently $\mu = -i$) for (2.3). We then transport this measurement to $s = i$ (equivalently $\mu = 0$). Thus, we will be needing the discretised versions of both (2.2) and (2.3).

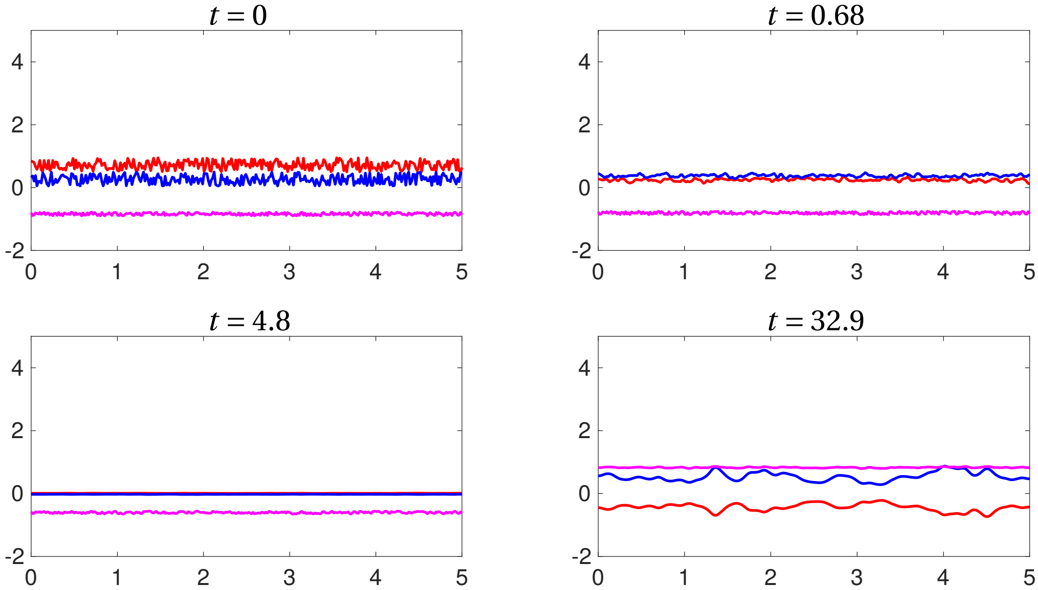


Figure 2.1: Numerical simulation of system (2.3). The red, blue and magenta lines denote $\mathrm{Re}z(x,t), \mathrm{Im}z(x,t)$ and $s(x,t)$ respectively at the given times t. We observe that the initial spatial inhomogeneities for $\mathrm{Re}z, \mathrm{Im}z$ rapidly smoothen and approach the constant zero function. After $s(x)$ becomes positive everywhere there is a delay until the instability can be seen, at around $s(x) \approx 1$.

Proposition 2.3. *The Galerkin discretisations of* (2.2) *and* (2.3), *truncated at* $k_0 \in \mathbb{N}$, *read*

$$z_1' = (2a)^{-1/2}(\mu_1 + i)z_1 + (2a)^{-1/2}\sum_{j=2}^{k_0}\mu_j z_j + \varepsilon\hat{g}_1(\mu_1,\dots,\mu_{k_0}), \tag{2.4a}$$

$$\mu_1' = (2a)^{1/2}\varepsilon, \tag{2.4b}$$

$$z_k' = \frac{b_k}{4a^2}z_k + (2a)^{-1/2}\big((\mu_1 + i)z_k + \mu_k z_1\big) + a^{-1/2}\sum_{n,j=2}^{k_0}\eta_{n,j}^k \mu_n z_j + \varepsilon\hat{g}_k(\mu_1,\dots,\mu_{k_0}), \tag{2.4c}$$

$$\mu_k' = \varepsilon\frac{b_k}{4a^2}\mu_k, \tag{2.4d}$$

and

$$z_1' = (2a)^{-1/2}s_1 z_1 + (2a)^{-1/2}\sum_{j=2}^{k_0}s_j z_j + \varepsilon\hat{h}_1(s_1,\dots,s_{k_0}), \tag{2.5a}$$

$$s_1' = (2a)^{1/2}\varepsilon, \tag{2.5b}$$

$$z_k' = \frac{b_k}{4a^2}z_k + (2a)^{-1/2}(s_1 z_k + s_k z_1) + a^{-1/2}\sum_{i,j=2}^{k_0}\eta_{i,j}^k s_i z_j + \varepsilon\hat{h}_k(s_1,\dots,s_{k_0}), \tag{2.5c}$$

$$s_k' = \varepsilon\frac{b_k}{4a^2}s_k, \tag{2.5d}$$

respectively where

$$\hat{h}_k(s_1,\dots,s_{k_0}) := \left\langle \hat{h}(s(x,t)), e_k(x)\right\rangle, \hat{g}_k(\mu_1,\dots,\mu_{k_0}) := \left\langle \hat{g}(s(x,t)), e_k(x)\right\rangle. \tag{2.6}$$

Here $\eta^k_{i,j} \in [0,1]$ *is defined by,*

$$\frac{1}{\sqrt{a}}\eta^k_{i,j} := \langle e_i e_j, e_k\rangle \tag{2.7}$$

and is non-zero if and only if $|i-j| = k-1$.

Proof. Considering $z = \sum_{k=1}^{\infty} z_k e_k$ and $s = \sum_{k=1}^{\infty} s_k e_k$, where the orthonormal basis is given by

$$e_{k+1}(x) = \sqrt{\frac{1}{a}}\cos\left(\frac{k\pi(x+a)}{2a}\right) \quad \text{and} \quad \lambda_{k+1} = -\frac{k^2\pi^2}{4a^2} \quad \text{for } k = 0,1,2,\dots, \tag{2.8}$$

projecting (2.3) on eigenfunctions e_k, and truncating at k_0 yields the system of ODEs (2.5). □

Remark 2.4. Observe that the translation $s = \mu + i$ in the original PDEs corresponds to setting $s_1 = \mu_1 + 1$ in the discrete systems.

2.2.2 Slow and Galerkin manifolds

In analogy to finite-dimensional singularly perturbed fast-slow systems, we begin our analysis by identifying a critical manifold and concern ourselves with the question of persistence of it. Before proceeding, we have to ensure that (2.1) is locally well-posed. This follows from standard results [4, 26] and the equation generates a semiflow on the space $X \times Y = L^2([-a,a]) \times L^2([-a,a])$.

In the slow time scale $\tau = \varepsilon t$, (2.3) becomes

$$\varepsilon z_\tau = z_{xx} + sz + \varepsilon \hat{h}(s), \tag{2.9a}$$

$$s_\tau = s_{xx} + 1. \tag{2.9b}$$

In the singular limit $\varepsilon \to 0$ we identify the critical manifold as the set

$$\{(z,s) : z_{xx} + sz = 0, z_x(\mp a) = 0 = s_x(\mp a)\}. \tag{2.10}$$

The usage of $\mp$ instead of $\pm$ is purely for aesthetic reasons. Restricting our attention to constant in space solutions, we identify the two components $\{z = 0\}$ and $\{s = 0\}$, with an abuse of notation identifying constant functions with the value they take. When $\varepsilon = 0$, the steady state at $z = 0$ is linearly stable when $s < 0$ and unstable when $s > 0$. Thus, we consider the two half-lines of steady states

$$S^a_0 := \{z = 0, s < 0\} \text{ and } S^r_0 := \{z = 0, s > 0\}. \tag{2.11}$$

In Proposition 2.5 we provide a form of persistence for the stable branch S^a_0; for S^r_0 we comment in subsection 2.2.9. The result is analogous to Proposition 1.5 and the discussion there transfers verbatim for the current system. Thus we refer there for more detail.

Proposition 2.5. *Let* $(0,s) \in S_0^a$ *so that* $s<0$. *Let* $\zeta>0$ *and* $\omega_A, \omega_f \in \mathbb{R}, L_f>0$ *such that* $s \le \omega_A < 0$ *and* $\omega_A + L_f < \omega_f < 0$. *Then, there exists a decomposition* $Y_S^\zeta \oplus Y_F^\zeta = L^2([-a,a])$ *and a family of slow manifolds around* $(0,s)$ *that are given as graphs*

$$S^a_{\varepsilon,\zeta} := \left\{ \left(f_X^{\varepsilon,\zeta}(s), f_{Y_F^\zeta}^{\varepsilon,\zeta}(s), s \right) : s \in Y_S^\zeta \right\} \tag{2.12}$$

for $0<\varepsilon<C\frac{\omega_f}{\omega_A}\zeta$ *and* $0<C<1$ *fixed, where* $\left(f_X^{\varepsilon,\zeta}(s), f_{Y_F^\zeta}^{\varepsilon,\zeta}(s), s \right) : Y_S^\zeta \to H^2([-a,a]) \times (Y_F^\zeta \cap H^2([-a,a]))$.

Proof. We have to show that the conditions of [30, Theorem 2.4] are satisfied. We refer to section 1.3, where the same procedure is carried out. We mention the most crucial point, which is the splitting $Y_S^\zeta \oplus Y_F^\zeta = Y$ that reveals the relation between ζ and k_0, in the sense that

$$Y_S^\zeta := \operatorname{span}\{e_k(x) : 0 \le k \le k_0\}, \text{ and} \tag{2.13a}$$

$$Y_F^\zeta := \overline{\operatorname{span}\{e_k(x) : k > k_0 a\}}^{L^2} \tag{2.13b}$$

where $\{e_k, \lambda_k\}$ are the eigenvalues and eigenfunctions (2.8) and

$$-\frac{(k_0+1)^2\pi^2}{4a^2} < \zeta^{-1}\omega_A \le -\frac{k_0^2\pi^2}{4a^2}. \tag{2.14}$$

□

These slow manifolds can be approximated by the so-called *Galerkin manifolds* $G_{\varepsilon,\zeta}$ of the truncated system (2.5). These Galerkin manifolds are precisely the Fenichel slow manifolds of (2.5) obtained by standard GSPT of the critical manifold $\{z_1 = z_k = s_k = 0, s_1 < 0\}$. Recall that each $\zeta > 0$ corresponds to a truncation level of $k_0 \approx \zeta^{-1/2}$. These are given by graphs

$$G_{\varepsilon,\zeta} := \left\{ \left(f_G^{\varepsilon,\zeta}(s), s \right) : s \in Y_S^\zeta \right\} \tag{2.15}$$

for a function $f_G^{\varepsilon,\zeta} : Y_S^\zeta \to X_S^\zeta$.

Proposition 2.6. *For* $0<\varepsilon<C\frac{\omega_f}{\omega_A}\zeta$ *where* $C, \zeta, \omega_A, \omega_f$ *are as in Proposition 2.5, the following estimate holds:*

$$\left\| f_X^{\varepsilon,\zeta}(s) - f_G^{\varepsilon,\zeta}(s) \right\|_{H^2} + \left\| f_{Y_F^\zeta}^{\varepsilon,\zeta}(s) \right\|_{H^2} \le \tilde{C} \left(\frac{4a^2}{\pi^2(2k_0+1)} + \zeta \right) \|s\|_{H^2}. \tag{2.16}$$

In particular, using the relation between ζ *and* k_0 *in* (2.14), *we have*

$$\left\| f_X^{\varepsilon,\zeta}(s) - f_G^{\varepsilon,\zeta}(s) \right\|_{H^2} + \left\| f_{Y_F^\zeta}^{\varepsilon,\zeta}(s) \right\|_{H^2} \le \tilde{C} \frac{1}{k_0} \|s\|_{H^2}. \tag{2.17}$$

Proof. See [39] and the application of that result in [18]. Here one also needs to split X in a fashion similar to Y, namely,

$$X_S^\zeta := \text{span}\{e_k(x) : 0 \leq k \leq k_0\}, \text{ and} \tag{2.18a}$$

$$X_F^\zeta := \overline{\text{span}\{e_k(x) : k > k_0 a\}}^{L^2} \tag{2.18b}$$

□

2.2.3 Fast-slow analysis

Without loss of generality, in order to simplify the calculations, we rescale the variables by $\hat{z}_j = (2a)^{-1/2} z_j, \hat{s}_j = (2a)^{-1/2} s_j$, for $2 \leq j \leq k_0$, and remove the $\hat{\square}$ notation to obtain

$$z_1' = s_1 z_1 + \sum_{j=2}^{k_0} s_j z_j + \varepsilon h_1(s_1, \ldots, s_{k_0}), \tag{2.19a}$$

$$s_1' = \varepsilon, \tag{2.19b}$$

$$z_k' = \frac{b_k}{4a^2} z_k + (s_1 z_k + s_k z_1) + 2^{1/2} \sum_{i,j=2}^{k_0} \eta_{i,j}^k s_i z_j + \varepsilon h_k(s_1, \ldots, s_{k_0}), \tag{2.19c}$$

$$s_k' = \varepsilon \frac{b_k}{4a^2} s_k, \tag{2.19d}$$

where now

$$h_k\left(s_1, \ldots, s_{k_0}\right) := (2a)^{-1/2} \hat{h}_k\left((2a)^{1/2} s_1, \ldots, (2a)^{1/2} s_{k_0}\right). \tag{2.20}$$

Similarly, we rewrite (2.4) as

$$z_1' = (\mu_1 + i) z_1 + \sum_{j=2}^{k_0} \mu_j z_j + \varepsilon g_1(\mu_1, \ldots, \mu_{k_0}), \tag{2.21a}$$

$$\mu_1' = \varepsilon, \tag{2.21b}$$

$$z_k' = \frac{b_k}{4a^2} z_k + \left((\mu_1 + i) z_k + \mu_k z_1\right) + 2^{1/2} \sum_{n,j=2}^{k_0} \eta_{n,j}^k \mu_n z_j + \varepsilon g_k(\mu_1, \ldots, \mu_{k_0}), \tag{2.21c}$$

$$\mu_k' = \varepsilon \frac{b_k}{4a^2} \mu_k. \tag{2.21d}$$

This is a fast-slow system, with z_k being the fast and s_k the slow variables. The reduced problem is

$$0 = s_1 z_1 + \sum_{j=2}^{k_0} s_j z_j \tag{2.22a}$$

$$\dot{s}_1 = 1, \tag{2.22b}$$

$$0 = \frac{b_k}{4a^2} z_k + (s_1 z_k + s_k z_1) + 2^{1/2} \sum_{i,j=2}^{k_0} \eta_{i,j}^k s_i z_j \tag{2.22c}$$

$$\dot{s}_k = \frac{b_k}{4a^2} s_k. \tag{2.22d}$$

The first task is to identify the critical manifold $\mathscr{C}$, given by solving the equations

$$0 = s_1 z_1 + \sum_{j=2}^{k_0} s_j z_j \tag{2.23a}$$

$$0 = \frac{b_k}{4a^2} z_k + (s_1 z_k + s_k z_1) + 2^{1/2} \sum_{i,j=2}^{k_0} \eta_{i,j}^k s_i z_j. \tag{2.23b}$$

for $(z_1, z_j, s_1, s_j), 2 \leq j \leq k_0$. Writing down the critical manifold $\mathscr{C}$ explicitly is not possible for arbitrary k_0, as it involves solving the linear system (2.23a) which is dense. However, we can immediately identify that the set

$$\mathscr{C}_0 := \{z_1 = 0, s_1 \in \mathbb{R}, z_k = s_k = 0\} \subset \mathscr{C} \tag{2.24}$$

is part of the critical manifold. Linearising the layer problem

$$z_1' = s_1 z_1 + \sum_{j=2}^{k_0} s_j z_j \tag{2.25a}$$

$$s_1' = 0, \tag{2.25b}$$

$$z_k' = \frac{b_k}{4a^2} z_k + (s_1 z_k + s_k z_1) + 2^{1/2} \sum_{i,j=2}^{k_0} \eta_{i,j}^k s_i z_j \tag{2.25c}$$

$$s_k' = 0, \tag{2.25d}$$

about the subset $\mathscr{C}_0$, we find the eigenvalues

$$\operatorname{diag}\left\{ s_1, s_1 + \frac{b_2}{4a^2}, \ldots, s_1 + \frac{b_{k_0}}{4a^2} \right\}; \tag{2.26}$$

thus $\mathscr{C}_0$ is normally hyperbolic attracting for $s_1 < 0$ and normally hyperbolic of saddle type for $0 < s_1 < -\frac{b_2}{4a^2}$. At $s_1 = 0$, the eigenvalue corresponding to the first mode changes sign and normal hyperbolicity is lost.

Remark 2.7. If we try to linearise around points of $\mathscr{C}$ with coordinates $s_j \neq 0, 2 \leq j \leq k_0$, the Jacobian matrix is dense and thus it is impossible to explicitly calculate the eigenvalues for such points with arbitrary k_0. We could do this for fixed values of k_0 though, as below where we examine the case $k_0 = 2$.

2.2.4 Example with $k_0 = 2$

It is perhaps useful to examine the discrete system when $k_0 = 2$ so that we can appreciate the complex geometry that arises even in the first non-trivial truncation level. Let us note that exactly this geometry and the additional singularities that neighbour the origin do not allow the reduction of the problem to the normal form treated in [25]. With $k_0 = 2$ and $a = \frac{\pi}{2}$, the system is

$$z_1' = s_1 z_1 + s_2 z_2 + \varepsilon h_1, \tag{2.27a}$$
$$s_1' = \varepsilon, \tag{2.27b}$$
$$z_2' = -z_2 + s_1 z_2 + s_2 z_1 + \varepsilon h_2, \tag{2.27c}$$
$$s_2' = -\varepsilon s_2. \tag{2.27d}$$

Splitting into real and imaginary parts by letting $z_j := u_j + i v_j$ we find

$$u_1' = s_1 u_1 + s_2 u_2 + \varepsilon \mathrm{Re} h_1, \tag{2.28a}$$
$$v_1' = s_1 v_1 + s_2 v_2 + \varepsilon \mathrm{Im} h_1, \tag{2.28b}$$
$$s_1' = \varepsilon, \tag{2.28c}$$
$$u_2' = -u_2 + s_1 u_2 + s_2 u_1 + \varepsilon \mathrm{Re} h_2, \tag{2.28d}$$
$$v_2' = -v_2 + s_1 v_2 + s_2 v_1 + \varepsilon \mathrm{Im} h_2, \tag{2.28e}$$
$$s_2' = -\varepsilon s_2. \tag{2.28f}$$

The critical manifold is given by solutions of the system

$$s_1 u_1 + s_2 u_2 = 0, \tag{2.29a}$$
$$s_1 v_1 + s_2 v_2 = 0, \tag{2.29b}$$
$$-u_2 + s_1 u_2 + s_2 u_1 = 0, \tag{2.29c}$$
$$-v_2 + s_1 v_2 + s_2 v_1 = 0 \tag{2.29d}$$

which we view as a linear system for (u_1, v_1, u_2, v_2). In matrix form,

$$J_2 \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = 0, \tag{2.30}$$

where

$$J_2 := \begin{pmatrix} s_1 & 0 & s_2 & 0 \\ 0 & s_1 & 0 & s_2 \\ s_2 & 0 & s_1 - 1 & 0 \\ 0 & s_2 & 0 & s_1 - 1 \end{pmatrix}. \tag{2.31}$$

The determinant

$$\det J_2 = \left(s_1 - s_1^2 + s_2^2\right)^2, \tag{2.32}$$

and consequently, if $0 < s_1 < 1$, the only solution to (2.30) is the trivial one; which allows us to give a description of the critical manifold for this range of s_1. For negative values of s_1, we have that $\det J_2 = 0$ for

$$s_1 = g(s_2) := \frac{1}{2}\left(1 - \sqrt{1 + 4 s_2^2}\right). \tag{2.33}$$

On this curve, $\operatorname{rank} J_2 = 2$ and non-trivial solutions of (2.31) exist. This provides a description of the critical manifold around the origin, which is the part of the phase

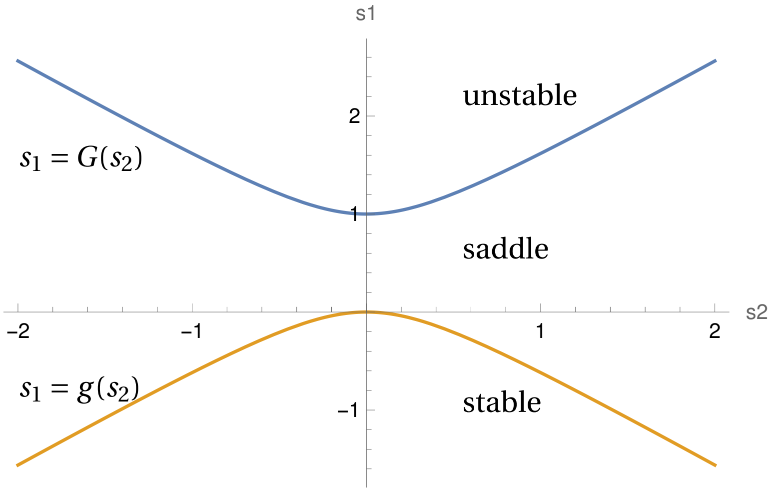


Figure 2.2: Stability of critical manifold $\mathscr{C}_0$ for $k_0 = 2$.

space we are interested in.

Lemma 2.8. *The points of the critical manifold* $\mathscr{C}$ *for* $|s_1| < 1$ *are exactly given by*

$$\mathscr{C} = \left\{(z_1, z_2, s_1, s_2) : z_1 = z_2 = 0, s_1 \in (0,1) \text{ or } s_1 < g(s_2), s_2 \in \mathbb{R}\right\} \cup \left\{(z_1, z_2, s_1, s_2) : z_1 \in \mathbb{C}, z_2 = \frac{2s_2}{1+\sqrt{1+4s_2^2}} z_1, s_1 = g(s_2), s_2 \in \mathbb{R}\right\}. \tag{2.34}$$

The linearisation of the layer problem has the two double eigenvalues

$$-\frac{1}{2} + s_1 \pm \frac{1}{2}\sqrt{1+4s_2^2}, \tag{2.35}$$

thus $\mathscr{C}$ *is normally hyperbolic attracting for* $s_1 < g(s_2)$*, normally hyperbolic of saddle type when* $s_1 > g(s_2)$ *as one of the eigenvalues switches along the curve* $s_1 = g(s_2)$ *and normally hyperbolic unstable as the second eigenvalue switches sign along the curve* $s_1 = G(s_2)$*; see Figure 2.2.*

Remark 2.9. The critical "manifold" $\mathscr{C}$ is not a manifold in this case. The term critical manifold is used in the hyperbolic theory where situations such as the one in Lemma 2.8 do not occur but has persisted in GSPT.

We see that the fast-slow structure of even the $k_0 = 2$ system is not straightforward and there is still much to be said about the dynamics around these two non-normally hyperbolic curves. For our analysis we focus only on the subset $\mathscr{C}_0$ of $\mathscr{C}$ and leave a full treatment for future work. In the general k_0 case, the normally hyperbolic parts of $\mathscr{C}$ will be separated by $k_0 - 1$-dimensional surfaces that make such a treatment challenging. A blowup of these surfaces will possibly reveal features of the PDE dynamics and allow insight into the double $\varepsilon \to 0, k \to \infty$ limit.

2.2.5 Main result

For general k_0, the geometry of the critical manifold will be similar, in the sense that it will consist at least the set $\mathscr{C}_0$, which, in turn, contains the two normally hyperbolic parts

$$\mathscr{C}_0^a := \left\{s_1 \in [-C_1, -c_1], s_j = 0\right\} \subset \mathscr{C}_0 \tag{2.36}$$

and

$$\mathscr{C}_0^r := \left\{s_1 \in [c_1, C_1], s_j = 0\right\} \subset \mathscr{C}_0, \tag{2.37}$$

where $c_1 < C_1$, $C_1 < \frac{|b_2|}{4a^2}$ and $C_j, 2 \le j \le k_0$ are small, positive constants that depend on k_0 only. These two subsets are of particular interest since they are exactly the critical manifold of the original PDE system defined in (2.11), projected onto the subspace of $L^2([-a, a])$ generated by the first k_0 eigenvalues $\{e_j(x)\}_{j=1}^{k_0}$.

As already mentioned, linearising about $\mathscr{C}_0^{a,r}$, we obtain the Jacobian matrix

$$\operatorname{diag}\left\{s_1, s_1 + \frac{b_2}{4a^2}, \ldots, s_1 + \frac{b_{k_0}}{4a^2}\right\} \tag{2.38}$$

which justifies the superscripts a, r respectively, since the first component is attracting, while the second on of saddle type, with one positive eigenvalue and $k_0 - 1$ negative.

Since the eigenvalues of a matrix depend continuously on the entries, there exist normally hyperbolic neighbourhoods around $\mathscr{C}_0^a, \mathscr{C}_0^r$, given as graphs over $s_1, s_j, 2 \le j \le k_0$, which will perturb, for $\varepsilon > 0$ to the sets

$$\begin{aligned} &\{(z_1, \ldots, z_{k_0}, s_1, \ldots, s_{k_0}) : (z_1, z_j) = F^a(s_1, s_j, \varepsilon), s_1 \in [-C_1, -c_1], |s_j| \le C_j\} \\ &\{(z_1, \ldots, z_{k_0}, s_1, \ldots, s_{k_0}) : (z_1, z_j) = F^r(s_1, s_j, \varepsilon), s_1 \in [c_1, C_1], |s_j| \le C_j\}, \end{aligned} \tag{2.39}$$

for some functions F^a, F^r and $C_j, 2 \le j \le k_0$ small constants, that depend on k_0. As already mentioned, $\mathscr{C}_0^a, \mathscr{C}_0^r$ are normally hyperbolic subsets, thus they also perturb to slow manifolds $\mathscr{C}_\varepsilon^a, \mathscr{C}_\varepsilon^r$ respectively described as

$$\mathscr{C}_\varepsilon^a = \{(z_1, z_j) = f^a(s_1, \varepsilon)\}, \tag{2.40a}$$

$$\mathscr{C}_\varepsilon^r = \{(z_1, z_j) = f^r(s_1, \varepsilon)\} \tag{2.40b}$$

where

$$f^a(s_1, \varepsilon) := F^a(s_1, 0, \varepsilon) \text{ and } f^r(s_1, \varepsilon) := F^r(s_1, 0, \varepsilon). \tag{2.41}$$

Remark 2.10. One has to be careful with (2.41); while there will be a part of the slow manifold above the hyperplane $\{s_1 \in [-C_1, c_1], s_j = 0\}$, obtained by evaluating F^a at $s_j = 0$, this does not immediately mean that this curve will be $\mathscr{C}_\varepsilon^a$. When viewed in isolation, there is no reason to expect $\mathscr{C}_0^a$ to perturb into an object whose $s_j, 2 \le j \le k_0$ coordinates are zero. However, this is the case for our our system of equations due to the slow variable dynamics being decoupled. Looking at the reduced problem (2.22), the set $\{s_1 \in [-C_1, c_1], s_j = 0\}$ is invariant and attracting for the slow dynamics. Since the dynamics on the slow manifold converge to the dynamics of the critical manifold, there will exist and invariant attracting set on the slow manifolds as well, which as we see from the full system has to be $\{s_j = 0\}$.

The main result for this section concerns the separation of these two perturbed manifolds at $s_1 = 0$:

Theorem 2.11. *The vector-valued separation between $\mathscr{C}_\varepsilon^a, \mathscr{C}_\varepsilon^r$ at $s_1 = 0$, defined by*

$$d(\varepsilon) := |f^a(0,\varepsilon) - f^r(0,\varepsilon)| = \left(|z_1^a - z_1^r|, |z_2^a - z_2^r|, \ldots, |z_{k_0}^a - z_{k_0}^r|\right)$$

satisfies

$$d(\varepsilon) = \left((2\pi)^{1/2}|h(0)|\varepsilon^{1/2} + \mathscr{O}(\varepsilon), \mathscr{O}(\varepsilon), \ldots, \mathscr{O}(\varepsilon)\right) \tag{2.42}$$

as $\varepsilon \to 0$.

2.2.6 Geometric desingularisation

To prove Theorem 2.11 we will perform a blowup of the origin of (2.19), which is where normal hyperbolicity of $\mathscr{C}_0$ is lost. Our goal is analyse orbits with initial conditions in the section

$$\Delta^{\text{in}} := \{|z_1| \le \beta, s_1 = -\rho, |z_k| \le C_{z_k}, |s_k| \le C_{s_k}\}, \tag{2.43}$$

$\beta, C_{z_k}, C_{s_k} > 0$ being appropriately chosen constants, as they pass near the singularity.

Remark 2.12. The origin is *not* fully nilpotent, the eigenvalues being 0 corresponding to z_1, s_1, s_k and $\frac{b_j}{4a^2}$ for $z_j, 2 \le j \le k_0$. Thus, to desingularise the origin via a blowup transformation we need to further process the system.

There approaches have been tried this far. In [19], the domain length a is included as a dummy variable and is blown up along with the rest. A disadvantage of this approach is that it results in a negative blow up weight for a leading to difficulties as the resulting vector field has a singularity for $a_i = 0$, where a_i is a in the three charts.

Instead, we elect to use again the idea introduced in [18] and apply a ε-dependent domain rescaling by introducing a new constant $A > 0$ through

$$a = A\varepsilon^p$$

resulting in the system

$$z_1' = s_1 z_1 + \sum_{j=2}^{k_0} s_j z_j + \varepsilon h_1, \tag{2.44a}$$

$$s_1' = \varepsilon, \tag{2.44b}$$

$$z_k' = \frac{b_k}{4A^2}\varepsilon^{-2p} z_k + (s_1 z_k + s_k z_1) + 2^{1/2}\sum_{i,j=2}^{k_0} \eta_{i,j}^k s_i z_j + \varepsilon h_k, \tag{2.44c}$$

$$s_k' = \frac{b_k}{4A^2}\varepsilon^{1-2p} s_k, \tag{2.44d}$$

Remark 2.13. This domain rescaling alters the fast-slow structure of the system and we have to keep this in mind when moving forward. Furthermore, the new vector field is not smooth in ε, but in $\varepsilon^{1/2}$ instead (as it turns out from the choice of p below); various object constructed, e.g. centre manifolds will thus be given as graphs of functions of $\varepsilon^{1/2}$ instead .

The origin of the new system is fully nilpotent as the only eigenvalue is 0. The power p will be chosen along the blowup weights so that the powers balance and we can desingularise the blown up vector fields. Consider the blow up transformation

$$z_k = \bar{r}^{\alpha_k} \bar{z}_k, \tag{2.45a}$$
$$s_k = \bar{r}^{\beta_k} \bar{s}_k, \tag{2.45b}$$
$$\varepsilon = \bar{r}^{\zeta} \bar{\varepsilon}. \tag{2.45c}$$

Replacing into (2.44), the smallest integers that satisfy the system and the resulting p are found to be

$$\alpha_1 = \alpha_k = 1, \quad \beta_1 = \beta_k = 1, \quad \zeta = 2, \quad p = -\frac{1}{4}. \tag{2.46}$$

We use the three charts formally given by

$$K_1 : \{\bar{s}_1 = -1\}, \quad K_2 : \{\bar{\varepsilon} = 1\}, \quad K_3 : \{\bar{s}_1 = 1\} \tag{2.47}$$

for our analysis. Orbits starting in K_1 and are attracted to $\mathcal{C}_\varepsilon^a$ will flow into K_2 forward in time and orbits starting in K_3 are are repelled by $\mathcal{C}_\varepsilon^r$ will flow in K_2 *backwards* in time. The separation measurement will be taken in K_2, followed by blow down to obtain the separation in terms of the original variables.

Lemma 2.14. *The change of coordinates between K_1 and K_2 is given by:*

$$\kappa_{12} : z_{1,2} = \varepsilon_1^{-1/2} z_{1,1}, s_{1,2} = -\varepsilon_1^{-1/2}, z_{k,2} = \varepsilon_1^{-1/2} z_{k,1}, s_{k,2} = \varepsilon_1^{-1/2} s_{k,1}, r_2 = \varepsilon_1^{1/2} r_1. \tag{2.48}$$

Proof. Direct calculation. □

2.2.7 Chart K_1

The transformation for this chart reads

$$z_1 = r_1 z_{1,1},\ s_1 = -r_1,\ z_k = r_1 z_{k,1},\ s_k = r_1 s_{k,1},\ \varepsilon = r_1^2 \varepsilon_1. \tag{2.49}$$

Using the transformation (2.49) and desingularising by a factor of r_1 we obtain

$$z_{1,1}' = -z_{1,1} + \varepsilon_1 z_{1,1} + \sum_{j=2}^{k_0} s_{j,1} z_{j,1} + \varepsilon_1 h_1\left(-r_1, r_1 s_{j,1}\right), \tag{2.50a}$$
$$r_1' = -r_1 \varepsilon_1, \tag{2.50b}$$
$$z_{k,1}' = \frac{b_k}{4A^2} \varepsilon_1^{1/2} z_{k,1} + \varepsilon_1 z_{k,1} + \left(-z_{k,1} + s_{k,1} z_{1,1}\right) + 2^{1/2} \sum_{i,j=2}^{k_0} \eta_{i,j}^k s_{j,1} z_{i,1} + \varepsilon_1 h_k(-r_1, r_1 s_{j,1}), \tag{2.50c}$$
$$s_{k,1}' = \frac{b_k}{4A^2} \varepsilon_1^{3/2} r_1^2 s_{k,1} + \varepsilon_1 s_{k,1}, \tag{2.50d}$$
$$\varepsilon_1' = 2\varepsilon_1^2, \tag{2.50e}$$

where abusing notation we write $h_k(-r_1, r_1 s_{j,1})$ for $h_k(-r_1, r_1 s_{2,1}, \ldots, r_1 s_{k_0,1})$. The vector field is not smooth in ε_1, thus we rewrite it in terms of $\varepsilon_1^{1/2}$ instead:

$$\left(\varepsilon_1^{1/2}\right)' = \left(\varepsilon_1^{1/2}\right)^3. \tag{2.51}$$

The subspaces $\{\varepsilon_1 = 0\}$ and $\{r_1 = 0\}$ are invariant and we have the line of steady states

$$\mathcal{S}_0^a := \{z_{1,1} = z_{k,1} = s_{k,1} = \varepsilon_1 = 0, r_1 \geq 0\},$$

parametrized by r_1. We can think of $\mathcal{C}_0^a$ as the corresponding object to $\mathcal{S}_0^a$ but not quite, since the introduction of the rescaling $A = a\varepsilon^{1/4}$ changes the fast-slow structure of the problem.

Remark 2.15. In addition to $\mathcal{S}_0^a$ there exist more steady states on the invariant subset $\{\varepsilon_1 = 0\}$ which for general k_0 cannot be described explicitly. These correspond to the parts of the critical manifold that lie outside $\mathcal{C}_0$. This is the reason we require ε-dependent initial values and the estimates for the higher order modes that are given below.

These steady states have eigenvalue -1 corresponding to the $z_{1,1}, z_{j,1}$ directions and 0 corresponding to the other variables. Therefore, an attracting $k_0 + 1$-dimensional center manifold, $W^{c,a}$ given as a graph

$$W_1^{c,a} = \left\{(z_{1,1}, z_{k,1}) = F_1\left(r_1, s_{j,1}, \varepsilon_1^{1/2}\right)\right\}. \tag{2.52}$$

emanates from $\mathcal{S}_0^a$. Since there exist more steady states around $\mathcal{S}_0^a$, $W^{c,a}$ is, more accurately, the subset of a larger attracting center manifold.

The “slices” of this graph for fixed small fixed ε_1 can be interpreted as the slow manifolds $\mathcal{C}_\varepsilon^a$ transformed to chart K_1, since the rescaling gives equivalent systems for $\varepsilon > 0$. These are defined as the graphs of F_1 for fixed $\varepsilon_1 \in [0,\delta]$:

$$\mathcal{S}_{\varepsilon_1}^a := \left\{(z_{1,1}, z_{j,1}) = F_1\left(r_1, 0, \varepsilon_1^{1/2}\right) : \varepsilon_1 \in [0,\delta] \text{ fixed}\right\}. \tag{2.53}$$

Next take the section

$$\Sigma_{1,k_0}^{\text{out}} := \{\varepsilon_1 = \delta\} \tag{2.54}$$

and

$$R_1^{\text{out}} := \Sigma_{1,k_0}^{\text{out}} \cap W^{c,a} = \Sigma_{1,k_0}^{\text{out}} \cap \mathcal{S}_\delta^a.$$

This will be the input to chart K_2, see Figure 2.3.

Example with $k_0 = 2$

Let us briefly examine the case $k_0 = 2$. The relevant system is

$$z_{1,1}' = -z_{1,1} + \varepsilon_1 z_{1,1} + s_{2,1} z_{2,1} + \varepsilon_1 h_1\left(-r_1, r_1 s_{2,1}\right), \tag{2.55a}$$

$$r_1' = -r_1 \varepsilon_1, \tag{2.55b}$$

$$z_{2,1}' = \frac{b_2}{4A^2}\varepsilon_1^{1/2} z_{2,1} + \varepsilon_1 z_{2,1} + \left(-z_{2,1} + s_{2,1} z_{1,1}\right) + \varepsilon_1 h_k(-r_1, -r_1 s_{2,1}), \tag{2.55c}$$

$$s_{2,1}' = \frac{b_2}{4A^2}\varepsilon_1^{3/2} r_1^2 s_{2,1} + \varepsilon_1 s_{2,1}, \tag{2.55d}$$

$$\varepsilon_1' = 2\varepsilon_1^2. \tag{2.55e}$$

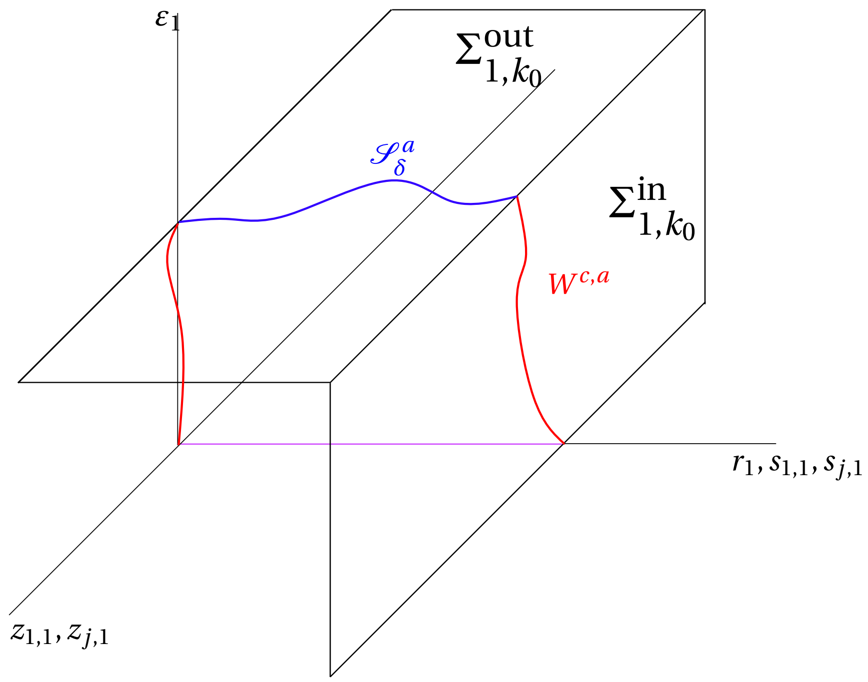


Figure 2.3: Dynamics in chart K_1.

The steady states of this system have $\varepsilon_1 = 0, r_1 \geq 0$ and

- either $z_{1,1} = 0$ in which case $s_{2,1} \in \mathbb{R}$ and we recover $\mathscr{S}^a_0$, or
- or $s_{2,1} = \pm 1$ in which case $z_{2,1} = \pm z_{1,1} \in \mathbb{R}$ which contains $\mathscr{S}^a_0$.

Estimates

The geometric description of the dynamics given above, while very useful in developing intuition, becomes complicated, especially when working with arbitrary k_0, and needs to be supplemented with additional analysis on the behaviour of the higher order modes. We adopt the same approach as in chapter 2, and provide some estimates based on the variation of constants formula. Advantages of this method are its simplicity and applicability to a large class of problems as is demonstrated in [29], where the estimates are given directly on the PDE level.

The essence of these estimates is to provide bounds for $z_{j,1}, s_{j,1}, 2 \leq j \leq k_0$ on R_1^{out}. As we shall see, the transition time $R^{\text{in}} \to \Sigma^{\text{out}}_{1,k_0}$ goes to infinity as $\varepsilon_1(0) \to 0$ and there is the possibility of numerical blowup of these variables. In turn, this facilitates the need to consider ε-dependent initial values. We begin by exploiting the simple form of (2.50e) and (2.50b).

Remark 2.16. In the following estimates we unroll A using the definition $A = a\varepsilon^{1/4} = ar(t)^{1/2}\varepsilon_1(t)^{1/4}$ to track all ε (or ε_1) factors.

Lemma 2.17. *The exact solutions of* (2.50e) *and* (2.50b) *are*

$$\varepsilon_1(t) = \frac{\varepsilon_1(0)}{1 - 2\varepsilon_1(0)\,t}, \tag{2.56a}$$

$$r_1(t) = \rho\,(1 - 2\varepsilon_1(0)\,t)^{1/2}. \tag{2.56b}$$

The transition time T_1, found by solving $\varepsilon_1(T_1) = \delta$, is

$$T_1 = \frac{1}{2}\left(\frac{1}{\varepsilon_1(0)} - \frac{1}{\delta}\right). \tag{2.57}$$

Proof. Direct integration. □

We proceed with estimating the higher order modes $z_{j,1}, s_{j,1}, 2 \le j \le k_0$.

Lemma 2.18. *For $2 \le j \le k_0$ and $t \in [0, T_1]$, it holds that*

$$s_{j,1}(t) = s_{j,1}(0)\frac{1}{(1-2\varepsilon_1(0)t)^{1/2}} \exp\left(\frac{b_j\rho}{4a^2}\left(1-(1-2\varepsilon_1(0)t)^{1/2}\right)\right). \tag{2.58}$$

In particular,

$$s_{j,1}(T_1) = s_{j,1}(0)\frac{\delta^{1/2}}{\varepsilon_1(0)^{1/2}} \exp\left(\rho\frac{b_j}{4a^2}\left(1-\frac{\varepsilon_1(0)^{1/2}}{\delta^{1/2}}\right)\right). \tag{2.59}$$

Proof. From the variation of constants formula,

$$s_{j,1}(t) = \exp\left(\int_0^t B_1(s)ds\right)s_{j,1}(0), \tag{2.60}$$

for $j = 2, \dots, k_0$, where, using the definition of A, we have

$$B_1(s) = \frac{b_j}{4A^2}\varepsilon_1(s)^{3/2}r_1^2(s) + \varepsilon_1(s) = \frac{b_j}{4a^2}\varepsilon_1(s)r_1(s) + \varepsilon_1(s).$$

Integration and use of the expression for T_1 in (2.57), yield

$$\begin{aligned} \exp\left(\int_0^t B_1(s)ds\right) &= \frac{1}{(1-2\varepsilon_1(0)t)^{1/2}} \exp\left(\rho\frac{b_j}{4a^2}\left(1-(1-2\varepsilon_1(0)t)^{1/2}\right)\right), \\ \exp\left(\int_0^{T_1} B_1(s)ds\right) &= \frac{\delta^{1/2}}{\varepsilon_1(0)^{1/2}} \exp\left(\rho\frac{b_j}{4a^2}\left(1-\frac{\varepsilon_1(0)^{1/2}}{\delta^{1/2}}\right)\right), \end{aligned} \tag{2.61}$$

which, combined with (2.60), implies the result stated in the lemma. □

Lemma 2.19. *Let*

$$\begin{aligned} Q_{i,j} &:= |s_{j,1}(0)| \sup_{s\in[0,T_1]} |z_{i,1}(s)|, \\ H_k &:= \sup_{s\in[0,T_1]} h_k\left(-r_1(s), r_1(s)s_{j,1}\right), \end{aligned}$$

for $1 \le i, j, k \le k_0$*.For* $z_1, z_k, 2 \le k \le k_0$ *and* $t \in [0, T_1]$*, the following estimates hold:*

$$
\begin{aligned}
|z_{1,1}(t)| &\le \frac{2}{e^{1/2}}|z_{1,1}(0)| + \sqrt{\frac{\delta}{\varepsilon_1(0)}}\sum_{j=1}^{k_0} Q_{j,j} + \delta H_1, \\
|z_{k,1}(t)| &\le \frac{2}{e^{1/2}}|z_{k,1}(0)| + Q_{1,k}\left(\frac{4a^2\rho}{|b_k|} + \frac{16a^2\rho^2\sqrt{\delta\varepsilon_1(0)}}{|b_k|^2}\right) \\
&\quad + \sum_{i,j=2}^{k_0} \eta^k_{i,j} Q_{i,j}\left(\frac{4a^2\rho}{|b_k|} + \frac{16a^4\sqrt{\delta\varepsilon_1(0)}\rho^2}{|b_k|^2}\right) \\
&\quad + H_k \frac{4a^2\rho\sqrt{\delta\varepsilon_1(0)}}{|b_k|}.
\end{aligned}
\tag{2.62}
$$

Proof. Denote

$$
\begin{aligned}
J_1(s) &:= -1 + \varepsilon_1(s), \\
N_1(s) &:= \sum_{j=2}^{k_0} s_{j,1}(s) z_{j,1}(s) + \varepsilon_1(s) h_1(-r_1(s), r_1(s)s_{j,1}(s))
\end{aligned}
$$

and

$$
J_k(s) := \frac{b_k}{4A^2}\varepsilon_1^{1/2}(s) + \varepsilon_1(s) - 1 = \frac{b_k}{4a^2}\frac{1}{r_1(s)} + \varepsilon_1(s) - 1, \tag{2.63}
$$

$$
N_k(s) := s_{k,1}(s)z_{1,1}(s) + 2^{1/2}\sum_{i,j=2}^{k_0} \eta^k_{i,j} s_{j,1}(s) z_{i,1}(s) + \varepsilon_1 h_k(-r_1(s), r_1(s)s_{j,1}(s)). \tag{2.64}
$$

First of all, for the linear part,

$$
\exp\left(\int_0^{T_1} J_k(s)ds\right) = \exp\left(\frac{1}{2\delta} - \frac{1}{2\varepsilon_1(0)}\right)\frac{\delta^{1/2}}{\varepsilon_1(0)^{1/2}}\exp\left(\frac{b_k}{4a^2\rho\varepsilon_1(0)}\left(1 - \frac{\varepsilon_1(0)^{1/2}}{\delta^{1/2}}\right)\right) \le 2e^{-1/2}, \tag{2.65}
$$

since $0 < \varepsilon_1(0) \le \delta \le 1$. Next, direct integration gives

$$
\begin{aligned}
\int_0^t J_k(\tau)d\tau &= -t - \frac{1}{2}\log(1 - 2\varepsilon_1(0)t) + \frac{b_k}{4a^2\rho\varepsilon_1(0)}\left(1 - \sqrt{1 - 2\varepsilon_1(0)t}\right), \\
\int_s^t J_k(\tau)d\tau &= (s - t) + \frac{1}{2}\left[\log(1 - 2\varepsilon_1(0)s) - \log(1 - 2\varepsilon_1(0)t)\right] \\
&\quad + \frac{b_k}{4a^2\rho\varepsilon_1(0)}\left(\sqrt{1 - 2\varepsilon_1(0)s} - \sqrt{1 - 2\varepsilon_1(0)t}\right),
\end{aligned}
$$

so that

$$
\exp\left(\int_0^t J_k(s)ds\right) = \frac{1}{\sqrt{1 - 2\varepsilon_1(0)t}}\exp\left(-t + \frac{b_k}{4a^2\rho}\frac{1 - \sqrt{1 - 2\varepsilon_1(0)t}}{\varepsilon_1(0)}\right) \tag{2.66}
$$

Let

$$\Phi_k(t,s) := \exp\left(\int_s^t J_k(\tau)d\tau\right) = \frac{1}{\sqrt{1-2\varepsilon_1(0)t}}\exp\left(-t-\frac{b_k}{4a^2\rho}\frac{\sqrt{1-2\varepsilon_1(0)t}}{\varepsilon_1(0)}\right)A(s), \tag{2.67}$$

where

$$A(s) := \sqrt{1-2\varepsilon_1(0)s}\exp\left(s+\frac{b_k}{4a^2\rho}\frac{\sqrt{1-2\varepsilon_1(0)s}}{\varepsilon_1(0)}\right).$$

Then, for $z_{1,1}$ we obtain

$$\begin{aligned}|z_{1,1}(t)| &\le \frac{|z_{1,1}(0)|e^{-t}}{\sqrt{1-2\varepsilon_1(0)t}} + \sum_{j=1}^{k_0} Q_{j,j}\frac{e^{-t}}{\sqrt{1-2\varepsilon_1(0)t}}\int_0^t \exp\left(s+\frac{b_j\rho}{4a^2}(1-\sqrt{1-2\varepsilon_1(0)s})\right)ds\\ &\quad + H_1\frac{\varepsilon_1(0)e^{-t}}{\sqrt{1-2\varepsilon_1(0)t}}\int_0^t \frac{e^s ds}{\sqrt{1-2\varepsilon_1(0)s}}\\ &\le 2e^{-\frac{1}{2}}|z_{1,1}(0)| + \sum_{j=1}^{k_0} Q_{j,j}\frac{\sqrt{\delta}}{\sqrt{\varepsilon_1(0)}} + \delta H_1.\end{aligned}$$

We now proceed with to derive estimates for $z_{k,1}$. We have to estimate three different terms in the non-autonomous part of the variation of constants formula in (2.64). Beginning with the sum term, for $t \in [0, T_1]$, we have

$$\begin{aligned}&\left|\int_0^t \Phi_k(t,s)\sum_{i,j=2}^{k_0}\eta_{i,j}^k s_{j,1}(s)z_{i,1}(s)ds\right| \le \sum_{i,j=2}^{k_0}\eta_{i,j}^k\frac{Q_{i,j}}{\sqrt{1-2\varepsilon_1(0)t}}\\ &\times\exp\left(-\frac{b_k}{4a^2\rho}\frac{\sqrt{1-2\varepsilon_1(0)t}}{\varepsilon_1(0)}\right)\int_0^t \exp\left(\frac{b_j\rho}{4a^2}(1-\sqrt{1-2\varepsilon_1(0)s})\right)\exp\left(\frac{b_k}{4a^2\rho}\frac{\sqrt{1-2\varepsilon_1(0)s}}{\varepsilon_1(0)}\right)ds\\ &\le \sum_{i,j=2}^{k_0}\eta_{i,j}^k\frac{Q_{i,j}}{\sqrt{1-2\varepsilon_1(0)t}}\exp\left(-\frac{b_k}{4a^2\rho}\frac{\sqrt{1-2\varepsilon_1(0)t}}{\varepsilon_1(0)}\right)\frac{1}{\varepsilon_1(0)}\int_{\sqrt{1-2\varepsilon_1(0)t}}^1 y\exp\left(\frac{b_k y}{4a^2\varepsilon_1(0)\rho}\right)dy\\ &= \sum_{i,j=2}^{k_0}\eta_{i,j}^k\frac{Q_{i,j}}{\sqrt{1-2\varepsilon_1(0)t}}\exp\left(-\frac{b_k}{4a^2\rho}\frac{\sqrt{1-2\varepsilon_1(0)t}}{\varepsilon_1(0)}\right)\times\\ &\quad\times\left(\frac{4a^2\rho}{b_k}\exp\left(\frac{b_k y}{4a^2\varepsilon_1(0)\rho}\right)y\Big|_{\sqrt{1-2\varepsilon_1(0)t}}^1 - \frac{4a^2\rho}{b_k}\int_{\sqrt{1-2\varepsilon_1(0)t}}^1\exp\left(\frac{b_k y}{4a^2\varepsilon_1(0)\rho}\right)dy\right)\\ &= \sum_{i,j=2}^{k_0}\eta_{i,j}^k\frac{Q_{i,j}}{\sqrt{1-2\varepsilon_1(0)t}}\exp\left(-\frac{b_k}{4a^2\rho}\frac{\sqrt{1-2\varepsilon_1(0)t}}{\varepsilon_1(0)}\right)\times\\ &\quad\times\left[\frac{4a^2\rho}{|b_k|}\exp\left(\frac{b_k}{4a^2\rho}\frac{\sqrt{1-2\varepsilon_1(0)t}}{\varepsilon_1(0)}\right)\sqrt{1-2\varepsilon_1(0)t} - \frac{4a^2\rho}{|b_k|}\exp\left(\frac{b_k}{4a^2\varepsilon_1(0)\rho}\right)\right.\\ &\qquad\left. - \frac{16a^4\varepsilon_1(0)\rho^2}{|b_k|^2}\exp\left(\frac{b_k}{4a^2\varepsilon_1(0)\rho}\right) + \frac{16a^4\varepsilon_1(0)\rho^2}{|b_k|^2}\exp\left(\frac{b_k}{4a^2\rho}\frac{\sqrt{1-2\varepsilon_1(0)t}}{\varepsilon_1(0)}\right)\right].\end{aligned}$$

We thus get the bound

$$\left|\int_0^t \Phi_k(t,s)\sum_{i,j=2}^{k_0}\eta_{i,j}^k s_{j,1}(s)z_{i,1}(s)ds\right| \le \sum_{i,j=2}^{k_0}\eta_{i,j}^k Q_{i,j}\left(\frac{4a^2\rho}{|b_k|} + \frac{16a^4\delta^{1/2}\varepsilon_1(0)^{1/2}\rho^2}{|b_k|^2}\right). \tag{2.68}$$

For the first term in (2.64),

$$
\begin{aligned}
&\left|\int_0^t \Phi_k(t,s)s_{k,1}(s)z_{1,1}(s)ds\right| \le \frac{Q_{1,k}}{\sqrt{1-2\varepsilon_1(0)t)}}\times\\
&\times\exp\Big(-\frac{b_k}{4a^2\rho}\frac{\sqrt{1-2\varepsilon_1(0)t}}{\varepsilon_1(0)}\Big)\int_0^t \exp\Big(\frac{b_k\rho}{4a^2}(1-\sqrt{1-2\varepsilon_1(0)s})\Big)\exp\Big(\frac{b_k}{4a^2\rho}\frac{\sqrt{1-2\varepsilon_1(0)s}}{\varepsilon_1(0)}\Big)ds\\
&\le \frac{Q_{1,k}}{\sqrt{1-2\varepsilon_1(0)t)}}\exp\Big(-\frac{b_k}{4a^2\rho}\frac{\sqrt{1-2\varepsilon_1(0)t}}{\varepsilon_1(0)}\Big)\int_{\sqrt{1-2\varepsilon_1(0)t)}}^1 \frac{y}{\varepsilon_1(0)}\exp\Big(\frac{b_k}{4a^2\rho}\frac{y}{\varepsilon_1(0)}\Big)dy\\
&\le Q_{1,k}\Big(\frac{4a^2\rho}{|b_k|}+\frac{16a^2\rho^2\varepsilon_1(0)}{\sqrt{1-2\varepsilon_1(0)t}|b_k|^2}\Big)\\
&\le Q_{1,k}\Big(\frac{4a^2\rho}{|b_k|}+\frac{16a^2\rho^2\sqrt{\varepsilon_1(0)}\sqrt{\delta}}{|b_k|^2}\Big),
\end{aligned}
$$

for all $t\in[0,T_1]$. We now turn out attention onto the last term in (2.64). We work as follows:

$$
\begin{aligned}
&\left|\int_0^t \Phi_k(t,s)\varepsilon_1(s)h_k(-r_1,r_1s_{j,1})ds\right|\\
&\le \frac{H_k}{\sqrt{1-2\varepsilon_1(0)t}}\exp\Bigg(-t-\frac{b_k}{4a^2\rho}\frac{\sqrt{1-2\varepsilon_1(0)t}}{\varepsilon_1(0)}\Bigg)\times\int_0^t\frac{\varepsilon_1(0)}{\sqrt{1-2\varepsilon_1(0)s}}\exp\Bigg(s+\frac{b_k}{4a^2\rho}\frac{\sqrt{1-2\varepsilon_1(0)s}}{\varepsilon_1(0)}\Bigg)ds\\
&\le \frac{H_k}{\sqrt{1-2\varepsilon_1(0)t}}\times\exp\Bigg(-\frac{b_k}{4a^2\rho}\frac{\sqrt{1-2\varepsilon_1(0)t}}{\varepsilon_1(0)}\Bigg)\int_{\sqrt{1-2\varepsilon_1(0)t}}^1\exp\Bigg(\frac{b_k}{4a^2\rho}\frac{y}{\varepsilon_1(0)}\Bigg)dy\\
&\le H_k\frac{\varepsilon_1(0)}{\sqrt{1-2\varepsilon_1(0)t}}\frac{4a^2\rho}{|b_k|}\\
&\le H_k\frac{4a^2\rho\sqrt{\delta}\sqrt{\varepsilon_1(0)}}{|b_k|},
\end{aligned}
$$

for all $t\in[0,T_1]$, and the proof is complete. □

2.2.8 Transition map

Let

$$R_1^{\text{in}} := \{z_{1,1}\in[-\beta,\beta],s_{1,1}=\rho,|z_{k,1}|\le C^{\text{in}}_{z_{k,1}}\varepsilon_1^{1/2},|s_{k,1}|\le C^{\text{in}}_{s_{k_1}}\varepsilon_1,\varepsilon_1\in[0,\delta]\}, \tag{2.69}$$

This is the set Δ^{in} transformed to the coordinates of K_1 and the constants that appear can be expressed in terms of the constants appear in the aforementioned definition. Combining the estimates from the previous section, we have the following result.

Proposition 2.20. *The transition map*

$$\Pi_1 : R_1^{\text{in}} \to \Sigma^{\text{out}}_{1,k_0} \tag{2.70}$$

is well-defined. For $(z^{\text{in}}_{1,1},s^{\text{in}}_{1,1},z^{\text{in}}_{k,1},s^{\text{in}}_{k,1},\varepsilon^{\text{in}})\in R_1^{\text{in}}$*, let*

$$(z^{\text{out}}_{1,1},s^{\text{out}}_{1,1},z^{\text{out}}_{k,1},s^{\text{out}}_{k,1},\delta)=\Pi_1(z^{\text{in}}_{1,1},s^{\text{in}}_{1,1},z^{\text{in}}_{k,1},s^{\text{in}}_{k,1},\varepsilon^{\text{in}}_1).$$

For $2 \le k \le k_0$, the following estimates hold:

$$|z_{1,1}^{\text{out}}| \le C_{z_{1,1}}^{\text{out}}, \tag{2.71}$$

$$|z_{k,1}^{\text{out}}| \le C_{z_{k,1}}^{\text{out}} (\varepsilon_1^{\text{in}})^{1/2}, \tag{2.72}$$

$$\left| s_{k,1}^{\text{out}} \right| \le C_{s_{k,1}}^{\text{out}} (\varepsilon_1^{\text{in}})^{1/2}, \tag{2.73}$$

Proof. For $s_{k,1}$, (2.73) follows directly from (2.59) and definition (2.69). To obtain estimates for $z_{k,1}^{\text{out}}$ we apply a fixed point argument in estimates in Lemma 2.19. Consider

$$\mathscr{M} = \{z_{j,1} \in C([0,T_1]) : |z_{1,1}(0)| \le C_{0,1},\ |z_{j,1}(0)| \le C_{0,j}\varepsilon_1(0)^{1/2},\ \sup_{[0,T]} |z_{1,1}(t)| \le C_1,$$
$$\sup_{[0,T_1]} |z_{j,1}(t)| \le C_j \varepsilon_1(0)^{1/2} \text{ with } \sum_{j=1}^{k_0} C_j \le \kappa\}$$

and $\tilde{z}_{j,1} \in \mathscr{M}$, for $j = 1, \dots k_0$, in the right-hand side of (2.62). Then choosing δ and ρ appropriate yields $z_{j,1} \in \mathscr{M}$, for $j = 1, \dots k_0$. □

2.2.9 Chart K_3

This system in this chart is similar to that in K_1 except for flipped signs in some terms of the equations for $z_{1,3}$ and $z_{k,3}$; compare with (2.50). After desingularising by a factor of of r_3,

$$z_{1,3}' = z_{1,3} - \varepsilon_3 z_{1,3} + \sum_{j=2}^{k_0} s_{j,3} z_{j,3} + \varepsilon_3 h_1 \left(r_3 s_{k,3} \right), \tag{2.74a}$$

$$r_3' = r_3 \varepsilon_3, \tag{2.74b}$$

$$z_{k,3}' = \frac{b_k}{4A^2} \varepsilon_3^{1/2} z_{k,3} - \varepsilon_3 z_{k,3} + \left(z_{k,3} + s_{k,3} z_{1,3} \right) + 2^{1/2} \sum_{i,j=2}^{k_0} \eta_{i,j}^k s_{j,3} z_{i,3} + \varepsilon_3 h_k (r_3 s_{j,3}), \tag{2.74c}$$

$$s_{k,3}' = \frac{b_k}{4A^2} \varepsilon_3^{3/2} r_3^2 s_{k,3} - \varepsilon_3 s_{k,3}, \tag{2.74d}$$

$$\varepsilon_3' = -2\varepsilon_3^2. \tag{2.74e}$$

The subspaces $\{\varepsilon_3 = 0\}$ and $\{r_3 = 0\}$ are once again invariant, and we have the line of steady states

$$\mathscr{S}_0^r := \{z_{1,3} = z_{k,3} = s_{k,3} = \varepsilon_3 = 0, r_3 \ge 0\} \tag{2.75}$$

parametrised by r_3. The comments on the corresponding steady states in chart K_1 apply here too, thus we refer to the discussion there. All the objects described there have their analogue in this chart too. These steady states have eigenvalue 1 corresponding to the $z_{1,3}, z_{j,3}$ directions and 0 for the other variables. Therefore, a repelling $k_0 + 1$-dimensional center manifold $W^{c,r}$ emanates from S_0^r. Under time reversal, this manifold becomes attracting and we can construct a transition map from

$$R_3^{\text{out}} := \{z_{1,3} \in [-\beta, \beta], s_{1,3} = \rho, |z_{k,3}| \leq C_{z_{k,3}}^{\text{out}} \varepsilon_3^p, |s_{k,3}| \leq C_{s_{k_3}}^{\text{out}} \varepsilon_3, \varepsilon_3 \in [0, \delta]\}, \tag{2.76}$$

to the section

$$\Sigma_{3,k_0}^{\text{in}} := \{\varepsilon_3 = \delta\}. \tag{2.77}$$

The following proposition then holds; the proof involves estimates such as those in chart K_1; since the systems differ only in some signs we do not repeat the statement and proof of these estimates and refer to subsection 2.2.7.

Proposition 2.21. *The transition map*

$$\Pi_3 : R_3^{\text{out}} \to \Sigma_{3,k_0}^{\text{in}} \tag{2.78}$$

is well-defined. For $(z_{1,3}^{\text{out}}, s_{1,3}^{\text{out}}, z_{k,3}^{\text{out}}, s_{k,3}^{\text{out}}, \varepsilon^{\text{out}}) \in R_3^{\text{out}}$*, let*

$$(z_{1,3}^{\text{in}}, s_{1,3}^{\text{in}}, z_{k,3}^{\text{in}}, s_{k,3}^{\text{in}}, \delta) = \Pi_3(z_{1,3}^{\text{out}}, s_{1,3}^{\text{out}}, z_{k,3}^{\text{out}}, s_{k,3}^{\text{out}}, \varepsilon_1^{\text{out}}).$$

For $2 \leq k \leq k_0$*, the following estimates hold:*

$$|z_{1,3}^{\text{in}}| \leq C_{z_{1,3}}^{\text{in}}, \tag{2.79}$$

$$|z_{k,3}^{\text{in}}| \leq C_{z_{k,3}}^{\text{in}} (\varepsilon_3^{\text{out}})^{1/2}, \tag{2.80}$$

$$\left| s_{k,3}^{\text{in}} \right| \leq C_{s_{k,3}}^{\text{in}} (\varepsilon_3^{\text{out}})^{1/2}. \tag{2.81}$$

Remark 2.22. We can interpret the backwards in time orbits as solutions of the final value problem for the original PDE (2.3), which is well-posed similarly to the initial value PDE.

2.2.10 Chart K_2

The system in K_2 after desingularising by a factor of r_2 is

$$z_{1,2}' = s_{1,2} z_{1,2} + \sum_{j=2}^{k_0} s_{j,2} z_{j,2} + h_1(r_2 s_{j,2}), \tag{2.82a}$$

$$s_{1,2}' = 1, \tag{2.82b}$$

$$z_{k,2}' = \frac{b_k}{4A^2} z_{k,2} + \left(s_{1,2} z_{k,2} + s_{k,2} z_{1,2}\right) + \sum_{i,j=2}^{k_0} \eta_{i,j}^k s_{i,2} z_{j,2} + h_k(r_2 s_{j,2}) \tag{2.82c}$$

$$s_{k,2}' = \frac{b_k}{4A^2} r_2^3 s_{k,2}, \tag{2.82d}$$

$$r_2' = 0. \tag{2.82e}$$

The initial conditions for this chart are a subset of the section

$$\Sigma_{2,k_0}^{\text{in}} := \left\{s_{1,2} = -\delta^{-1/2}\right\}. \tag{2.83}$$

More precisely, we have the following lemma.

Lemma 2.23. *The set*

$$R_2^{\text{in}} := \kappa_{12} \circ \Pi_{1,k_0}\left(R_1^{\text{in}}\right) \tag{2.84}$$

is described by

$$R_2^{\text{in}} = \Big\{ |z_{1,2}| \le \frac{1}{\delta^{1/2}\rho} C_{z_{1,1}}^{\text{out}}, s_{1,2} = -\delta^{-1/2}\,, \\ |z_{k,2}| \le \frac{1}{\delta^{1/2}\rho} C_{z_{1,1}}^{\text{out}} r_2, |s_{k,2}| \le \frac{1}{\delta^{1/2}\rho} C_{s_{1,1}}^{\text{out}} r_2, r_2 \in [0, \delta^{1/2}\rho] \Big\} \tag{2.85}$$

Proof. From the change of coordinates formula (2.48), we find that

$$r_2 = \varepsilon_1^{1/2} r_1 = (\varepsilon_1^{\text{in}})^{1/2}\rho.$$

Combining this with Proposition 2.20 and the change of coordinates (2.48), (2.85) follows by a straightforward calculation. □

The center manifold W_a^c is transformed by κ_{12} to a graph of a function F_2:

$$W_2^{c,a} = \left\{(z_{1,2}, z_{j,2}) = F_2^a(r_2, s_{1,2}, s_{j,2})\right\}. \tag{2.86}$$

Thus we have the description of R_2^{in} as the graph

$$R_2^{\text{in}} = \left\{\delta^{-1/2} F_2^a(r_2, -\delta^{1/2}, s_{j,2})\right\}. \tag{2.87}$$

Similarly to K_1, we also have the transformed versions of $\mathscr{C}_\varepsilon^a$ in this chart

$$\mathscr{S}_{r_2,2}^a = \left\{\left(z_{1,2}, z_{j,2}\right) = f_2^a(r_2, s_{1,2})\right\}, \tag{2.88}$$

where $f_2^a(r_2, s_{1,2}) := F_2^a(r_2, s_{1,2}, 0)$. These have a similar interpretation as the corresponding objects in K_1; namely transformed slow manifolds of the system (2.19) in chart K_2 (with the usual caveat that the rescaling we do with A and a changes the fast-slow structure of the problem). We want to calculate

$$(z_{1,2}^a, z_{j,2}^a) = f_2^a(r_2, 0) \tag{2.89}$$

for small $r_2 > 0$; in other words we aim to find $(z_{1,2}^a, z_{j,2}^a)$ when an orbit starting on R_2^{in} crosses the $s_{1,2} = 0$ axis. This point is well-defined because in (2.82) we have $s_{1,2}' = 1$ and $s_{1,2}(0) = -\delta^{-1/2} < 0$. In addition, $|s_{k,2}(t)|$ is non increasing as $\frac{b_k}{4A^2} r_2^3 \le 0$ and we show that $|z_{k,2}(t)|$ remains bounded to, so that no finite-time blowup occurs.

Setting $r_2 = 0$,

$$z_{1,2}' = s_{1,2} z_{1,2} + \sum_{j=2}^{k_0} s_{j,2} z_{j,2} + h_1(0), \tag{2.90a}$$

$$s_{1,2}' = 1, \tag{2.90b}$$

$$z_{k,2}' = \frac{b_k}{4A^2} z_{k,2} + \left(s_{1,2} z_{k,2} + s_{k,2} z_{1,2}\right) + \sum_{i,j=2}^{k_0} \eta_{i,j}^k s_{i,2} z_{j,2} \tag{2.90c}$$

$$s'_{k,2} = 0, \tag{2.90d}$$
$$r'_2 = 0. \tag{2.90e}$$

Note we can assume $h_k(0) = 0$ without loss of generality. From Lemma 2.23 we have $|s_{j,2}(0)| = |z_{k,2}(0)| = \mathcal{O}(r_2)$. Taking s_k zero for $2 \le k \le k_0$ and linearising at $z_{k,2} = 0$ we find the factor

$$\left(\frac{b_k}{4A^2} + s_{1,2}\right) z_{k,2}.$$

Since we are tracking orbits for $s_{1,2} \in [-\delta^{-1/2}, 0]$, $z_{k,2} = 0$ is an exponentially stable equilibrium for $s_{1,2}$ in that range. Thus, $|z_{k,2}(t)|$ is non-increasing.

Remark 2.24. We could also use estimates like those in K_1 to obtain the same result.

We conclude that we can reduce the original system, by restricting r_2, to a regular perturbation of

$$z'_{1,2} = s_{1,2} z_{1,2} + h_1(0), \tag{2.91a}$$
$$s'_{1,2} = 1. \tag{2.91b}$$

Taking $s_{1,2}$ as the independent variable,

$$\frac{\mathrm{d}z_{1,2}}{\mathrm{d}s_{1,2}} = s_{1,2} z_{1,2} + h_1(0) \tag{2.92}$$

and using an integrating factor,

$$\left(e^{-\frac{s_{1,2}^2}{2}} z_2\right)' = h_1(0) e^{-\frac{s_{1,2}^2}{2}} \tag{2.93}$$

Using that this singular orbit is asymptotic to the origin in K_1, that is $z_{1,2}(s_{1,2} = -\infty) = 0$, we get

$$z_{1,2}(s_{1,2} = 0) = h_1(0) \int_{-\infty}^{0} e^{-\frac{s_{1,2}^2}{2}}\, ds = -h_1(0)\sqrt{\frac{\pi}{2}}, \tag{2.94}$$

which implies that

$$f_2^a(0,0) = \left(\mp h_1(0)\left(\frac{\pi}{2}\right)^{1/2} + \mathcal{O}(r_2), \mathcal{O}(r_2)\right). \tag{2.95}$$

By regular perturbation theory, small $r_2 > 0$ simply induces an $\mathcal{O}(r_2)$ correction.

We also need to calculate

$$(z^r_{1,2}, z^r_{j,2}) = f_2^r(r_2, 0), \tag{2.96}$$

by the tracking the intersections of orbits that enter the chart at $\Sigma^{\text{out}}_{2,k_0} := \{s_{1,2} = \delta^{-1/2}$ from chart K_3 backwards in time with the $\{s_{1,2} = 0\}$ hyperplane; all the objects with the $\square^a$ superscript have the corresponding $\square^r$ object and everything stated above holds for these orbits too. We have then proven the main proposition of this section.

Proposition 2.25. *It holds that*

$$(z^a_{1,2}, z^a_{j,2}) := f^a_2(r_2, 0) = \left(-h_1(0)\left(\frac{\pi}{2}\right)^{1/2} + \mathcal{O}(r_2), \mathcal{O}(r_2)\right), \tag{2.97}$$

$$(z^r_{1,2}, z^r_{j,2}) := f^r_2(r_2, 0) = \left(h_1(0)\left(\frac{\pi}{2}\right)^{1/2} + \mathcal{O}(r_2), \mathcal{O}(r_2)\right). \tag{2.98}$$

Consequently,

$$|f^a_2(r_2, 0) - f^r_2(r_2, 0)| = \left((2\pi)^{1/2}|h(0)| + \mathcal{O}(r_2), \mathcal{O}(r_2)\right). \tag{2.99}$$

As a corollary, we have the proof of our main result.

Proof of Theorem 2.11. Undoing the blowup transformation, for $r_2 > 0$ sufficiently small, we have $r_2 = \varepsilon^{1/2}$ and $z_j = r_2 z_{j,2} = \varepsilon^{1/2} z_{j,2}$ on K_2. It follows immediately from (2.99) that

$$d(\varepsilon) = \left((2\pi)^{1/2}|h(0)|\varepsilon^{1/2} + \mathcal{O}(\varepsilon), \mathcal{O}(\varepsilon), \ldots, \mathcal{O}(\varepsilon)\right). \tag{2.100}$$

□

We finish by showing that the separation of $\mathcal{C}^a_\varepsilon, \mathcal{C}^r_\varepsilon$ is exponentially small in ε for (2.21). Let

$$w(\mu_1) = (w_1(\mu_1), \ldots, w_{k_0}(\mu_1)) := f^a(\mu_1, \varepsilon) - f^r(\mu_1, \varepsilon) \tag{2.101}$$

Proposition 2.26. *It holds that* $w(0) = d(\varepsilon)e^{-\frac{1}{2\varepsilon}}$.

Proof. We have established $|w(-i)| = d(\varepsilon)$, which under the translation $s_1 = \mu_1 + i$ corresponds to $s_1 = 0$ and have to calculate $|w(0)|$. Using initial values on $\mathcal{C}^a_\varepsilon, \mathcal{C}^r_\varepsilon$ respectively in system (2.19), we see that w satisfies

$$w_1' = (\mu_1 + i)w_1, \tag{2.102a}$$
$$\mu_1' = \varepsilon, \tag{2.102b}$$
$$w_k' = \frac{b_k}{4a^2}w_k + (\mu_1 + i)w_k, \tag{2.102c}$$

as points on the two aforementioned curves lie the $\{s_j = 0, 2 \le j \le k_0\}$ hyperplane. Equivalently,

$$\frac{\mathrm{d}w_1}{\mathrm{d}\mu_1} = \frac{1}{\varepsilon}\left(\mu_1 + i\right)w_1, \tag{2.103a}$$
$$\frac{\mathrm{d}w_k}{\mathrm{d}\mu_1} = \frac{1}{\varepsilon}\left(\frac{b_k}{4a^2} + \mu_1 + i\right)w_k., \tag{2.103b}$$

and integrating along the imaginary axis from $\mu_1 = -i$ up to $\mu_1 = 0$, we find

$$w_1(0) = w_1(-i)e^{-\frac{1}{2\varepsilon}}, \quad w_k(0) = w_k(-i)e^{-\frac{1}{2\varepsilon} + \frac{1}{\varepsilon}\frac{b_k}{4a^2}i}, \tag{2.104}$$

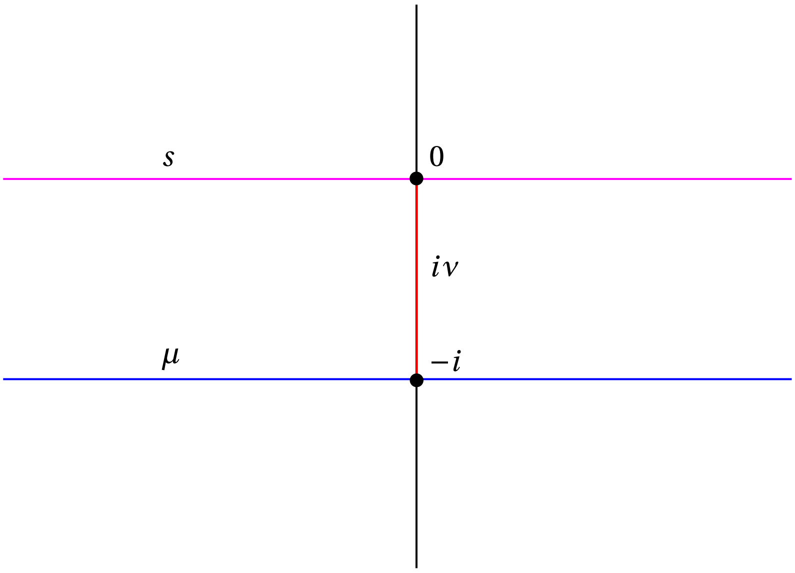


Figure 2.4: The relation between the variables s, μ, v in the complex plane.

therefore,

$$|w(0)| = |w(-i)|e^{-\frac{1}{2\varepsilon}} = d(\varepsilon)e^{-\frac{1}{2\varepsilon}}. \tag{2.105}$$

□

2.3 Full Hopf system

We now turn our attention to the original problem (2.1). Most of the work done for the PDE version of the Shishkova system, such as the fast-slow analysis, the geometry and the blowup transformation can be carried over to this case almost verbatim while the most challenging part is adapting the estimates in chart K_1 taking into account the new third order terms in the resulting vector field.

Recall system (2.1); under the same transformation $\mu := s - i$ as in system (2.3) we have

$$z_t = z_{xx} + sz - |z|^2 z + \varepsilon h_0 \tag{2.106a}$$

$$s_t = \varepsilon(s_{xx} + 1), \tag{2.106b}$$

for some $h_0 \neq 0$. Due to the presence of the third order terms, to write down the discretised system, we need split $z_1, z_j, 2 \leq j \leq k_0$ intro real and imaginary parts; let

$$z_j = u_j + iv_j. \tag{2.107}$$

Then, the truncated at k_0 Galerkin discretisation is

$$z_1' = (2a)^{-1/2} s_1 z_1 - \frac{2}{a} z_1 |z_1|^2 + (2a)^{-1/2} \varepsilon \hat{h}_0 + (2a)^{-1/2} \sum_{j=2}^{k_0} s_j z_j \tag{2.108a}$$

$$- \frac{1}{2a} z_1 \sum_{j=2}^{k_0} |z_j|^2 - \frac{1}{2^{1/2} a} \sum_{n,j,l=2}^{k_0} \eta^l_{n,j} (u_n u_j + v_n v_j) z_l,$$

$$s_1' = (2a)^{1/2} \varepsilon, \tag{2.108b}$$

$$z_k' = \frac{b_k}{4a^2} z_k + (2a)^{-1/2} (s_1 z_k + s_k z_1) + a^{-1/2} \sum_{n,j=2}^{k_0} \eta^k_{n,j} s_n z_j \tag{2.108c}$$

$$+ \frac{1}{2a} \sum_{n,j,l=1}^{k_0} \theta^k_{n,j,l} \left(u_n u_j + v_n v_j\right) z_l,$$

$$s_k' = \varepsilon \frac{b_k}{4a^2} s_k. \tag{2.108d}$$

The real numbers $\theta^k_{n,j,l} \in [0,2]$ are defined by

$$\langle e_n e_j e_l, e_k \rangle = \frac{1}{2a} \theta^k_{n,j,l} \tag{2.109}$$

so that

$$\begin{aligned} \theta^k_{n,j,l} = &\int_0^1 \cos[(n+j+l-3)\pi x] \cos[(k-1)\pi x]\, dx \\ &+ \int_0^1 \cos[(n+j-l-1)\pi x] \cos[(k-1)\pi x]\, dx \\ &+ \int_0^1 \cos[(n-j+l-1)\pi x] \cos[(k-1)\pi x]\, dx \\ &+ \int_0^1 \cos[(n-j-l+1)\pi x] \cos[(k-1)\pi x]\, dx. \end{aligned} \tag{2.110}$$

It follows that $\theta^k_{n,j,l} \neq 0$ if and only if at least one of the following holds:

$$\begin{aligned} n+j+l-3 &= k-1, \\ n+j-l-1 &= k-1, \\ n-j+l-1 &= k-1, \\ n-j-l+1 &= k-1. \end{aligned}$$

Consequently, if we fix any three of k, n, j, l, only four choices of the remaining index will give non-zero $\theta^k_{n,j,l}$. We also observe that

$$\theta^k_{1,j,l} = 2\eta^k_{j,l}. \tag{2.111}$$

As we did for the Shishkova system, we rescale the variables in (2.108) by $\hat{z}_k = (2a)^{-1/2} z_k$, $\hat{s}_k = (2a)^{-1/2} s_k$ while we also set $h_0 := (2a)^{-1/2} \hat{h}_0$. This gives, after removing the $\hat{\square}$ notation,

$$z_1' = s_1 z_1 - 4 z_1 |z_1|^2 + \varepsilon h_0 + \sum_{j=2}^{k_0} s_j z_j - z_1 \sum_{j=2}^{k_0} |z_j|^2 - 2^{1/2} \sum_{n,j,l=2}^{k_0} \eta^l_{n,j} (u_n u_j + v_n v_j) z_l, \tag{2.112a}$$

$$s_1' = \varepsilon, \tag{2.112b}$$

$$z_k' = \frac{b_k}{4a^2} z_k + (s_1 z_k + s_k z_1) + \sum_{n,j=2}^{k_0} \eta_{n,j}^k s_n z_j + 2|z_1|^2 z_k + 4 z_1 (u_1 u_k + v_1 v_k) \quad (2.112c)$$

$$+ 2 z_1 \sum_{n,j=2}^{k_0} \eta_{n,j}^k (u_n u_j + v_n v_j) + \sum_{n,j,l=2}^{k_0} \theta_{n,j,l}^k \left(u_n u_j + v_n v_j\right) z_l,$$

$$s_k' = \varepsilon \frac{b_k}{4a^2} s_k, \quad (2.112d)$$

which is equivalent, and the system we will be working with onwards. We have also separated the terms that contains z_1 in the last sum of the equation for z_k.

Remark 2.27. When we give the equations for systems in the following, we will be using the compact or separated form of the sum of third order terms in the z_k interchangeably, depending on the importance of having the first mode clearly visible.

The definition of the parts of the critical manifold $\mathscr{C}_0^a, \mathscr{C}_0^r$ along with all the related objects, can be carried over, as well as all the comments on the geometry of this fast-slow system. The main result of this section is the following:

Theorem 2.28. *The separation between* $\mathscr{C}_0^a, \mathscr{C}_0^r$ *at* $s_1 = 0$ *defined by*

$$d(\varepsilon) := |f^a(0,\varepsilon) - f^r(0,\varepsilon)| = \left(|z_1^a - z_1^r|, |z_2^a - z_2^r|, \ldots, |z_{k_0}^a - z_{k_0}^r|\right)$$

satisfies

$$d(\varepsilon) = \left((2\pi)^{1/2} |h_0| C \varepsilon^{1/2} + \mathscr{O}(\varepsilon), \mathscr{O}(\varepsilon), \ldots, \mathscr{O}(\varepsilon)\right) \quad (2.113)$$

as $\varepsilon \to 0$, *for a positive constant* $C > 0$.

The origin once again is a non-hyperbolic steady state with negative or zero eigenvalues, necessitating the same blowup transformation combined with domain rescaling

$$z_k = \bar{r} \bar{z}_k, \quad s_k = \bar{r} \bar{s}_k, \quad \varepsilon = \bar{r}^2 \bar{\varepsilon}, \quad A = a \varepsilon^{1/4}. \quad (2.114)$$

We will be making use of three charts, namely

$$K_1 : \{\bar{s}_1 = -1\}, \quad K_2 : \{\bar{\varepsilon} = 1\}, \quad K_3 : \{\bar{s}_1 = 1\}. \quad (2.115)$$

2.3.1 Chart K_1

The transformation for this chart is

$$z_1 = r_1 z_{1,1}, \quad s_1 = -r_1, \quad z_k = r_1 z_{k,1}, \quad s_k = r_1 s_{k,1}, \quad \varepsilon = r_1^2 \varepsilon_1 \quad (2.116)$$

and after desingularising by a factor of r_1, the system is

$$z_{1,1}' = \varepsilon_1 z_{1,1} - z_{1,1} - 4 r_1 z_{1,1} |z_{1,1}|^2 + \varepsilon_1 h_0 + \sum_{j=2}^{k_0} s_{j,1} z_{j,1} - r_1 z_{1,1} \sum_{j=2}^{k_0} |z_{j,1}|^2 \quad (2.117a)$$

$$- 2^{1/2} r_1 \sum_{n,j,l=2}^{k_0} \eta_{n,j}^l (u_{n,1} u_{j,1} + v_{n,1} v_{j,1}) z_{l,1},$$

$$r_1' = -r_1\varepsilon_1, \tag{2.117b}$$

$$z_{k,1}' = \varepsilon_1 z_{k,1} + \frac{b_k}{4A^2}\varepsilon_1^{1/2} z_{k,1} + \left(-z_{k,1} + s_{k,1} z_{1,1}\right) + \sum_{n,j=2}^{k_0} \eta_{n,j}^k s_{n,1} z_{j,1} \tag{2.117c}$$

$$+ r_1 \sum_{n,j,l=1}^{k_0} \theta_{n,j,l}^k \left(u_{n,1}u_{j,1} + v_{n,1}v_{j,1}\right) z_{l,1},$$

$$s_{k,1}' = \frac{b_k}{4A^2}\varepsilon_1^{3/2} r_1^2 s_{k,1} + \varepsilon_1 s_{k,1}, \tag{2.117d}$$

$$\varepsilon_1' = 2\varepsilon_1^2. \tag{2.117e}$$

The geometry remains largely the same because the third order terms have r_1 in front of them and vanish on the $r_1 = 0$ plane. As mentioned, the main challenge in extending the results from the Shishkova system to the full non-linear Hopf bifurcation is obtaining new estimates taking into account the new third order terms. Once again, we can begin by directly solving for $r_1(t), \varepsilon_1(t)$ and calculating the transition time T_1. This is a repeat of Lemma 2.17: The exact solutions of (2.117b) and (2.117e) are

$$\varepsilon_1(t) = \frac{\varepsilon_1(0)}{1 - 2\varepsilon_1(0)t}, \tag{2.118a}$$

$$r_1(t) = \rho\,(1 - 2\varepsilon_1(0)t)^{1/2}. \tag{2.118b}$$

The transition time T_1 is found by solving $\varepsilon_1(T_1) = \delta$ and is given by

$$T_1 = \frac{1}{2}\left(\frac{1}{\varepsilon_1(0)} - \frac{1}{\delta}\right). \tag{2.119}$$

The required property for the transition map is the same as the one in Proposition 2.20. The definition of the in and out sections remain the same for this system, i.e.

$$R_1^{\text{in}} := \{z_{1,1} \in [-\beta,\beta], s_{1,1} = \rho, |z_{k,1}| \le C_{z_{k,1}}^{\text{in}}\varepsilon_1^{1/2}, |s_{k,1}| \le C_{s_{k_1}}^{\text{in}}\varepsilon_1, \varepsilon_1 \in [0,\delta]\}, \tag{2.120}$$

and $\Sigma_{1,k_0}^{\text{out}} := \{\varepsilon_1 = \delta\}$.

Proposition 2.29. *The transition map*

$$\Pi_1 : R_1^{\text{in}} \to \Sigma_{1,k_0}^{\text{out}} \tag{2.121}$$

is well-defined. For $(z_{1,1}^{\text{in}}, s_{1,1}^{\text{in}}, z_{k,1}^{\text{in}}, s_{k,1}^{\text{in}}, \varepsilon^{\text{in}}) \in R_1^{\text{in}}$*, let*

$$(z_{1,1}^{\text{out}}, s_{1,1}^{\text{out}}, z_{k,1}^{\text{out}}, s_{k,1}^{\text{out}}, \delta) = \Pi_1(z_{1,1}^{\text{in}}, s_{1,1}^{\text{in}}, z_{k,1}^{\text{in}}, s_{k,1}^{\text{in}}, \varepsilon_1^{\text{in}}).$$

For $2 \le k \le k_0$*, the following estimates hold:*

$$|z_{1,1}^{\text{out}}| \le C_{z_{1,1}}^{\text{out}}, \tag{2.122}$$

$$|z_{k,1}^{\text{out}}| \le C_{z_{k,1}}^{\text{out}}(\varepsilon_1^{\text{in}})^{1/2}, \tag{2.123}$$

$$\left|s_{k,1}^{\text{out}}\right| \le C_{s_{k,1}}^{\text{out}}(\varepsilon_1^{\text{in}})^{1/2}, \tag{2.124}$$

Proof. To prove the proposition for the full Hopf system, once again we perform a series of elementary estimates using the variation of constants formula. Since the equations for $s_{k,1}$ remain the same, Lemma 2.18 holds in the present case too without any modification. In the way as in the previous section, for $2 \leq j \leq k_0$ and $t \in [0, T_1]$, we obtain

$$s_{j,1}(t) = \frac{s_{j,1}(0)}{(1-2\varepsilon_1(0)t)^{1/2}} \exp\left(\frac{b_j\rho}{4a^2}\left(1-(1-2\varepsilon_1(0)t)^{1/2}\right)\right). \tag{2.125}$$

In particular,

$$s_{j,1}(T_1) = s_{j,1}(0)\frac{\delta^{1/2}}{\varepsilon_1(0)^{1/2}} \exp\left(\frac{b_j\rho}{4a^2}\left(1-\frac{\varepsilon_1(0)^{1/2}}{\delta^{1/2}}\right)\right). \tag{2.126}$$

To derive estimate for $z_{1,1}$ we compute

$$\begin{aligned} J_1(t) &= \exp\left\{\int_0^t \left(\varepsilon_1 - 1 - 4r_1|z_1|^2 - r_1\sum_{j=2}^{k_0}|z_{j,1}|^2\right)d\tau\right\} \\ &\leq \exp\left\{\int_0^t (\varepsilon_1(\tau)-1)\,d\tau\right\} \\ &= \frac{e^{-t}}{\sqrt{1-2\varepsilon_1(0)t}}. \end{aligned}$$

Then

$$\begin{aligned} |z_{1,1}(t)| \leq & \frac{|z_{1,1}(0)|\,e^{-t}}{\sqrt{1-2\varepsilon_1(0)t}} + \sum_{j=2}^{k_0}\sup_{[0,T_1]}|z_{j,1}|\frac{|s_{j,1}(0)|e^{-t}}{\sqrt{1-2\varepsilon_1(0)t}}\int_0^t \exp\left(s+\frac{b_j\rho}{4a^2}(1-\sqrt{1-2\varepsilon_1(0)s})\right)ds \\ & + \sup_{[0,T_1]}|h_1|\frac{\varepsilon_1(0)e^{-t}}{\sqrt{1-2\varepsilon_1(0)t}}\int_0^t \frac{e^s\,ds}{\sqrt{1-2\varepsilon_1(0)s}} \\ & + 2^{1/2}\rho\frac{e^{-t}}{\sqrt{1-2\varepsilon_1(0)t}}\sum_{n,j,l=2}^{k_0}\eta^l_{n,j}\sup_{[0,T_1]}|z_{n,1}||z_{j,1}||z_{l,1}|\int_0^t e^s(1-2\varepsilon_1(0)s)\,ds \\ \leq & 2e^{-\frac{1}{2}}|z_{1,1}(0)| + \delta\sup_{[0,T_1]}|h_1| + \frac{\sqrt{\delta}}{\sqrt{\varepsilon_1(0)}}\sum_{j=2}^{k_0}|s_{j,1}(0)|\sup_{[0,T_1]}|z_{j,1}| \\ & + 2^{1/2}\rho\left(\frac{\sqrt{\varepsilon_1(0)}}{\sqrt{\delta}} + \sqrt{\varepsilon_1(0)\delta}\right)\sum_{n,j,l=2}^{k_0}\eta^l_{n,j}\sup_{[0,T_1]}|z_{n,1}||z_{j,1}||z_{l,1}|, \end{aligned} \tag{2.127}$$

for all $t \in [0, T_1]$. For $z_{k,1}$, we work similarly to section 2.2.7. Letting

$$\Phi_k(t,s) := \exp\left(\int_s^t J_k(\tau)\,d\tau\right) = \frac{1}{\sqrt{1-2\varepsilon_1(0)t}}\exp\left(-t - \frac{b_k}{4a^2\rho}\frac{\sqrt{1-2\varepsilon_1(0)t}}{\varepsilon_1(0)}\right)A(s), \tag{2.128}$$

with

$$J_k(s) := \frac{b_k}{4A^2}\varepsilon_1^{1/2}(s) + \varepsilon_1(s) - 1 = \frac{b_k}{4a^2}\frac{1}{r_1(s)} + \varepsilon_1(s) - 1$$

and

$$A(s) := \sqrt{1-2\varepsilon_1(0)s}\exp\left(s + \frac{b_k}{4a^2\rho}\frac{\sqrt{1-2\varepsilon_1(0)s}}{\varepsilon_1(0)}\right),$$

the terms $s_{k,1}z_{1,1}, \sum_{n,j=2}^{k_0}\eta^k_{n,j}s_{n,1}z_{j,1}$ are estimated in the same way as before; we need only to estimate only the last term in (2.117c). To this end, for all $t \in [0, T_1]$, we have

$$\left|\int_0^t \Phi_k(t,s) r_1(s) \sum_{n,j,l=1}^{k_0} \theta_{n,j,l}^k \left(u_{n,1}u_{j,1} + v_{n,1}v_{j,1}\right) z_{l,1} ds\right|$$
$$\leq \rho \sum_{n,j,l=1}^{k_0} \theta_{n,j,l}^k \sup_{[0,T_1]} |z_{n,1}||z_{j,1}||z_{l,1}| \frac{1}{\sqrt{1-2\varepsilon_1(0)t}} \times$$
$$\times \exp\left(-\frac{b_k}{4a^2\rho}\frac{\sqrt{1-2\varepsilon_1(0)t}}{\varepsilon_1(0)}\right)\int_0^t (1-2\varepsilon_1(0)s)\exp\left(\frac{b_k}{4a^2\rho}\frac{\sqrt{1-2\varepsilon_1(0)s}}{\varepsilon_1(0)}\right) ds$$
$$\leq \sum_{n=1,j=1,l=2}^{k_0} \theta_{n,j,l}^k \sup_{[0,T_1]} |z_{n,1}||z_{j,1}||z_{l,1}| \frac{C\rho^2}{|b_k|}(1-2\varepsilon_1(0)t)$$
$$\leq C \sum_{n,j=1,l=2}^{k_0} \theta_{n,j,l}^k \sup_{[0,T_1]} |z_{n,1}||z_{j,1}||z_{l,1}| \frac{\rho^2}{|b_k|}\frac{\varepsilon_1(0)}{\delta}.$$

Putting everything together, for $z_{k,1}, 2 \leq k \leq k_0$, we obtain

$$\begin{aligned}|z_{k,1}(t)| \leq &\frac{2}{e^{1/2}}|z_{k,1}(0)| + \sup_{[0,T_1]} |z_{1,1}(s)||s_{k,1}(0)|\left(\frac{4a^2\rho}{|b_k|} + \frac{16a^2\rho^2\sqrt{\delta\varepsilon_1(0)}}{|b_k|^2}\right)\\ &+ \sum_{i,j=2}^{k_0} \eta_{i,j}^k \sup_{[0,T_1]} |z_{i,1}(s)||s_{j,1}(0)|\left(\frac{4a^2\rho}{|b_k|} + \frac{16a^4\sqrt{\delta\varepsilon_1(0)}\rho^2}{|b_k|^2}\right)\\ &+ C\frac{\rho^2}{|b_k|}\frac{\varepsilon_1(0)}{\delta}\sum_{n,j=1,l=2}^{k_0} \theta_{n,j,l}^k \sup_{[0,T_1]} |z_{n,1}||z_{j,1}||z_{l,1}|.\end{aligned} \tag{2.129}$$

Since $|s_{j,1}(0)| \leq C_{s_{j,1}}^{\text{in}}\varepsilon_1(0)$, we have $\sup_{[0,T_1]}|s_{j,1}| \leq C_{s_{j,1}}^{\text{out}}\varepsilon_1(0)^{1/2}, 2 \leq j \leq k_0$. As before, we finish the proof by combining the obtained estimates with a fixed point argument. Define

$$\mathscr{M} := \left\{ z_{k,1} \in C([0,T_1]) : \sup_{t\in[0,T_1]} |z_{k,1}(t)| \leq C_k\varepsilon_1(0)^{1/2}, \text{ with } \sum_{n,j=2}^{k_0} C_n^2 C_j \leq \kappa \right\}.$$

Using $|z_{1,1}(0)| \leq \beta$ and $\hat{z}_{k,1} \in \mathscr{M}, 2 \leq k \leq k_0$, instead of $z_{k,1}$ in the right hand side of (2.127) we obtain $\sup_{[0,T_1]}|z_{1,1}| \leq C_{z_{1,1}}^{\text{out}}$. For $|z_{k,1}(0)| \leq C_{z_{k,1}}^{\text{in}}\varepsilon_1(0)^{1/2}$, considering $\hat{z}_{k,1} \in \mathscr{M}$ instead of $z_{k,1}$, for $k = 2,\ldots,k_0$, on the right-hand side of estimate (2.129) and applying the fixed-point argument we obtain estimates (2.123) and (2.122). □

2.3.2 Chart K_2

We again desingularise by a factor of r_2 to obtain the system in this chart:

$$\begin{aligned}z_{1,2}' = &s_{1,2}z_{1,2} - 4r_2 z_{1,2}|z_{1,2}|^2 + h_0 + \sum_{j=2}^{k_0} s_{j,2}z_{j,2} - r_2 z_{1,2}\sum_{j=2}^{k_0}|z_{j,2}|^2\\ &- 2^{1/2}r_2\sum_{n,j,l=2}^{k_0}\eta_{n,j}^l(u_{n,2}u_{j,2} + v_{n,2}v_{j,2})z_{l,2}\end{aligned} \tag{2.130a}$$

$$s'_{1,2} = 1, \tag{2.130b}$$

$$z'_{k,2} = \frac{b_k}{4A^2} z_{k,2} + (s_{1,2} z_{k,2} + s_{k,2} z_{1,2}) + \sum_{n,j=2}^{k_0} \eta^k_{n,j} s_n z_j \tag{2.130c}$$

$$+ r_2 \sum_{n,j,l=1}^{k_0} \theta^k_{n,j,l} (u_{n,2} u_{j,2} + v_{n,2} v_{j,2}) z_{l,2},$$

$$s'_{k,2} = \frac{b_k}{4A^2} r_2^3 s_{k,2}, \tag{2.130d}$$

$$r'_2 = 0. \tag{2.130e}$$

Once again, the new thirds order terms that are present are multiplied by r_2; thus when considering $r_2 = 0$ they vanish. We can then use the same argument as in subsection 2.2.10 to obtain that

$$z_{1,2}(s_{1,2} = 0) = -h_0 \sqrt{\frac{\pi}{2}}. \tag{2.131}$$

If $r_2 > 0$ is small, the third order terms induce further regular perturbations, thus we have the following statement:

Proposition 2.30. *It holds that*

$$(z^a_{1,2}, z^a_{j,2}) := f^a_2(r_2, 0) = \left(-h_0 \left(\frac{\pi}{2}\right)^{1/2} + \mathcal{O}(r_2), \mathcal{O}(r_2) \right), \tag{2.132}$$

$$(z^r_{1,2}, z^r_{j,2}) := f^r_2(r_2, 0) = \left(h_0 \left(\frac{\pi}{2}\right)^{1/2} + \mathcal{O}(r_2), \mathcal{O}(r_2) \right). \tag{2.133}$$

Consequently,

$$|f^a_2(r_2, 0) - f^r_2(r_2, 0)| = \left((2\pi)^{1/2} |h_0| + \mathcal{O}(r_2), \mathcal{O}(r_2) \right), \tag{2.134}$$

which proves the main result Theorem 2.28 after blowing down.

2.3.3 Distance measurement transport

We finish by transporting this distance measurement at $-i$ in the original variable μ, to $\mu = 0$. Let

$$w(\mu_1) = (w_1(\mu_1), \ldots, w_{k_0}(\mu_1)) := f^a(\mu_1, \varepsilon) - f^r(\mu_1, \varepsilon). \tag{2.135}$$

Proposition 2.31. *It holds that* $w(0) = d(\varepsilon) e^{-\frac{1}{2\varepsilon}}$.

Due to the presence of the third order nonlinear terms in (2.112) we cannot follow the same approach as for the Shishkova system. Instead we will again employ geometric desingularisation along with a series of estimates. Since we are tracking orbits of (2.112) starting on $f^a(\mu_1, \varepsilon), f^r(\mu_1, \varepsilon)$ which are a subset of the invariant subspace $\{s_j = 0 : 2 \leq j \leq k_0\}$, the system simplifies to

$$z_1' = s_1 z_1 - 4z_1|z_1|^2 + \varepsilon h_0 - z_1 \sum_{j=2}^{k_0} |z_j|^2 - 2^{1/2} \sum_{n,j,l=2}^{k_0} \eta_{n,j}^l (u_n u_j + v_n v_j) z_l, \tag{2.136a}$$

$$s_1' = \varepsilon, \tag{2.136b}$$

$$z_k' = \left(\frac{b_k}{4a^2} + s_1\right) z_k + \sum_{n,j,l=1}^{k_0} \theta_{n,j,l}^k \left(u_n u_j + v_n v_j\right) z_l. \tag{2.136c}$$

We wish to move the separation measurement distance from $\mu_1 = -i$ to $\mu_1 = 0$; equivalently from $s_1 = 0$ to $s_1 = i$. For convenience, introduce v_1 (Figure 2.4) by

$$s_1 = \mu_1 + i = i v_1, \tag{2.137}$$

so that now we track v_1 along the real segment from $v_1 = 0$ to $v_1 = 1$. Since $s_1' = \varepsilon$,

$$v_1' = -i\varepsilon \tag{2.138}$$

with $v_1(0) = 0$ as our initial condition. It follows that $v_1(t) = -i\varepsilon t$; if we calculate out exit time $\hat{T}$ by requiring $v_1(\hat{T}) = 1$ we see that $\hat{T} = \frac{1}{\varepsilon} i$ or in other words we need integrate with respect our independent variable t along a segment of the imaginary axis. It is more convenient to work with a real independent variable, so we conduct the change of time $\tau = -it$, where now $\tau \in \mathbb{R}$. Letting the overhead dot denote differentiation with respect to τ,

$$\dot{z}_1 = -v_1 z_1 + i\varepsilon h_0 - i \sum_{n,j,l=1}^{k_0} \eta_{n,j}^l (u_n u_j + v_n v_j) z_l, \tag{2.139a}$$

$$\dot{v}_1 = \varepsilon, \tag{2.139b}$$

$$\dot{z}_k = \left(i\frac{b_k}{4a^2} - v_1\right) z_k + i \sum_{n,j,l=1}^{k_0} \theta_{n,j,l}^k \left(u_n u_j + v_n v_j\right) z_l \tag{2.139c}$$

$$\dot{\varepsilon} = 0 \tag{2.139d}$$

In this alternate formulation, $v_1(\tau) = \varepsilon\tau$ with $v_1(0) = 0$ and $v_1(T) = 1$ where the new exit time $T = \frac{1}{\varepsilon}$.

Remark 2.32. In the corresponding analysis for transporting the distance measurement in [25], the change of time conducted it $\tau = it$. This is inconsequential in that work, since the estimates are performed by taking v as the independent variable throughout. While we do the same for chart K_2 below, we use the exit chart K_3 where we perform estimates using the time variable τ. Had we performed the same change of time here, we would have to work backwards in time, from $\tau = 0$ to $\tau = -T$ which would not have been as clear.

The origin at $(z_1, v_1, z_k, \varepsilon) = (0,0,0,0)$ of (2.139) is a non-hyperbolic steady state with a triple zero eigenvalue for z_1, v_1, ε and $-i\frac{b_j}{4a^2}$ for $z_j, 2 \leq j \leq k_0$. To analyse the dynamics near the origin, we employ the same rescaling-blowup transformation as before, namely,

$$z_1 = r\bar{z}_1, \quad v_1 = r\bar{v}_1, \quad z_k = r\bar{z}_k, \quad \varepsilon = r^2\bar{\varepsilon}, \quad a = A\varepsilon^{-1/4} \tag{2.140}$$

which turns the origin into a fully degenerate steady state:

$$\dot{z}_1 = -v_1 z_1 + i\varepsilon h_0 - i\sum_{n,j,l=1}^{k_0} \eta_{n,j}^l (u_n u_j + v_n v_j) z_l, \tag{2.141a}$$

$$\dot{v}_1 = \varepsilon, \tag{2.141b}$$

$$\dot{z}_k = \left(i\frac{b_k}{4A^2}\varepsilon^{1/2} - v_1\right) z_k + i\sum_{n,j,l=1}^{k_0} \theta_{n,j,l}^k \left(u_n u_j + v_n v_j\right) z_l \tag{2.141c}$$

$$\dot{\left(\varepsilon^{1/2}\right)} = 0. \tag{2.141d}$$

Remark 2.33. Systems (2.139) and (2.141) are equivalent for $\varepsilon > 0$.

We require two charts to track the orbits; the rescaling chart $K_2 : \{\bar{\varepsilon} = 1\}$ and the exit chart $K_3 : \{\bar{v}_1 = 1\}$. The same names were also used for charts in the Shishkova and Hopf systems earlier. From this point K_1, K_2 will refer to charts obtained by blowing up (2.141).

Chart K_2

In K_2, the transformation is

$$z_1 = r_2 z_{1,2}, \quad v_1 = r_2 v_{1,2}, \quad z_k = r_2 z_{k,2}, \quad \varepsilon = r_2^2 \tag{2.142}$$

and the corresponding system, after desingularising by a factor of r_2 becomes

$$\dot{z}_{1,2} = -v_{1,2} z_{1,2} + i h_0 - i r_2 \sum_{n,j,l=1}^{k_0} \eta_{n,j}^l (u_{n,2} u_{j,2} + v_{n,2} v_{j,2}) z_{l,2}, \tag{2.143a}$$

$$\dot{v}_{1,2} = 1, \tag{2.143b}$$

$$\dot{z}_{k,2} = \left(i\frac{b_k}{4A^2} - v_{1,2}\right) z_{k,2} + i r_2 \sum_{n,j,l=1}^{k_0} \theta_{n,j,l}^k (u_{n,2} u_{j,2} + v_{n,2} v_{j,2}) z_{l,2}, \tag{2.143c}$$

$$\dot{r}_2 = 0. \tag{2.143d}$$

We first study the dynamics on the invariant plane $\{r_2 = 0\}$; small $r_2 > 0$ will induce regular perturbations and will simply add an $\mathcal{O}(r_2)$ correction to our findings. Setting $r_2 = 0$,

$$\dot{z}_{1,2} = -v_{1,2} z_{1,2} + i h_0, \tag{2.144a}$$

$$\dot{v}_{1,2} = 1, \tag{2.144b}$$

$$\dot{z}_{k,2} = \left(i\frac{b_k}{4A^2} - v_{1,2}\right) z_{k,2} \tag{2.144c}$$

and taking $v_{1,2}$ as the independent variable we arrive at the explicit solutions

$$z_{1,2}(v_{1,2}) = z_{1,2}(v_{1,2}=0)e^{-v_{1,2}^2/2} + ih_0 e^{-v_{1,2}^2/2}\int_0^{v_{1,2}} e^{\sigma^2/2}d\sigma, \tag{2.145a}$$

$$z_{k,2}(v_{1,2}) = z_{k,2}(v_{1,2}=0)e^{i\frac{b_k}{4A^2}v_{1,2}+\frac{v_{1,2}^2}{2}}. \tag{2.145b}$$

Our initial conditions $z_{1,2}(v_{1,2}=0), z_{j,2}(v_{1,2}=0)$ are given by $f_2^{a,r}(0,0)$ (Proposition 2.30), i.e. $f^{a,r}(0,0)$ transformed in K_2; recall that we defined $s_{1,2} = iv_{1,2}$ so that $v_{1,2}$ corresponds to $s_{1,2}=0$. Let $z_1^{a,r}(\tau;\varepsilon), z_j^{a,r}(\tau;\varepsilon)$ denote the orbits with initial conditions $f^{a,r}(0,\varepsilon)$ of (2.139) respectively (and the subscripts the corresponding orbits in charts K_2, K_3). From (2.145) and, as mentioned, regular perturbation theory,

$$z_{1,2}^{a,r}(v_{1,2};r_2) = \left(\mp h_0\sqrt{\frac{\pi}{2}}+\mathcal{O}(r_2)\right)e^{-v_{1,2}^2/2} + ih_0e^{-v_{1,2}^2/2}\int_0^{v_{1,2}} e^{\sigma^2/2}d\sigma + \mathcal{O}(r_2), \tag{2.146a}$$

$$z_{k,2}^{a,r}(v_{1,2};r_2) = \mathcal{O}(r_2)e^{i\frac{b_k}{4A^2}v_{1,2}+\frac{v_{1,2}^2}{2}}, \tag{2.146b}$$

for any finite $v_{1,2}$. Consequently, the following proposition is proved.

Proposition 2.34. *On the exit section*

$$\Sigma_{2,k_0}^{\text{out}} := \{v_{1,2} = \xi\} \tag{2.147}$$

where $\xi > 0$ *is an arbitrarily chosen constant, we have*

$$z_{1,2}^{a,r}(\xi;r_2) = \mp h_0\sqrt{\frac{\pi}{2}}e^{-\xi^2/2} + ih_0e^{-\xi^2/2}\int_0^{\xi} e^{\sigma^2/2}d\sigma + \mathcal{O}(r_2), \tag{2.148a}$$

$$\left|z_{k,2}^{a,r}(\xi;r_2)\right| = \mathcal{O}(r_2). \tag{2.148b}$$

These will be our initial values in the subsequent analysis in the exit chart K_3.

Remark 2.35. Our analysis in K_2 follows closely the one found in [25]. There, instead of working in the exit chart K_3, the orbits coming out of K_2 are tracked in the original system. We elect to work in K_3 as it allows us to employ estimates such as those in K_1 once again.

Chart K_3

In K_3, the transformation and systems respectively are

$$z_1 = r_3 z_{1,3}, \quad v_1 = r_3, \quad z_k = r_3 z_{k,3}, \quad \varepsilon = r_3^2\varepsilon_3 \tag{2.149}$$

and

$$\dot{z}_{1,3} = -z_{1,3} - \varepsilon_3 z_{1,3} + ih_0 - ir_3\sum_{n,j,l=1}^{k_0}\eta_{n,j}^l(u_{n,3}u_{j,3}+v_{n,3}v_{j,3})z_{l,3}, \tag{2.150a}$$

$$\dot{r}_3 = r_3\varepsilon_3, \tag{2.150b}$$

$$\dot{z}_{k,3} = \left(i\frac{b_k}{4A^2}\varepsilon_3^{1/2} - 1 - \varepsilon_3\right) z_{k,3} + i r_3 \sum_{n,j,l=1}^{k_0} \theta_{n,j,l}^{k} (u_{n,3}u_{j,3} + v_{n,3}v_{j,3}) z_{l,3}, \tag{2.150c}$$

$$\dot{\varepsilon}_3 = -2\varepsilon_3^2. \tag{2.150d}$$

We will also need the change of coordinates $\kappa_{23} : K_2 \to K_3$

$$z_{1,3} = v_{1,2}^{-1} z_{1,2}, \quad r_3 = r_2 v_{1,2}, \quad z_{k,3} = v_{1,2}^{-1} z_{k,2}, \quad \varepsilon_3 = v_{1,2}^{-2}, \tag{2.151}$$

as well as the entry and exit sections

$$\Sigma_{3,k_0}^{\text{in}} := \kappa_{23}\left(\Sigma_{2,k_0}^{\text{out}}\right) = \{\varepsilon_3 = \xi^{-2}\}, \quad \Sigma_{3,k_0}^{\text{out}} := \{r_3 = 1\}. \tag{2.152}$$

We have the exact solutions

$$\varepsilon_3(\tau) = \frac{1}{2\tau + \xi^2}, \quad r_3(\tau) = \xi^{-1} r_3(0)(2\tau + \xi^2)^{1/2}, \tag{2.153}$$

and solving for the transition by T_3 the equation $r_3(T_3) = 1$, which corresponds to $v = 1$ as can by seen through (2.151), we find

$$T_3 = \frac{1}{2}\xi^2\left(\frac{1}{r_3^2(0)} - 1\right). \tag{2.154}$$

Using the relation between ε in (2.149) and $r_2, r_3(0)$ in charts K_2, K_3 respectively, it holds that $r_3(0) = \xi r_2 = \xi\varepsilon^{1/2}$ so that $\mathcal{O}(r_2) = \mathcal{O}(\varepsilon^{1/2})$. The initial values of the rest of the variables on $\Sigma_{3,k_0}^{\text{in}}$ are

$$z_{1,3}^{a,r}(0) = \mp\frac{1}{\xi} h_0 \sqrt{\frac{\pi}{2}} e^{-\xi^2/2} + i\frac{1}{\xi} h_0 e^{-\xi^2/2} \int_0^{\xi} e^{\sigma^2/2} d\sigma + \mathcal{O}(r_3(0)), \tag{2.155a}$$

$$z_{k,3}^{a,r}(0) = \mathcal{O}(r_3(0)). \tag{2.155b}$$

In particular, $z_{1,3}^{a}(0) - z_{1,3}^{r}(0) = -\frac{1}{\xi} h_0 \sqrt{2\pi} e^{-\xi^2/2} + \mathcal{O}(r_3(0))$.

Estimates in chart K_3

We now derive estimates for system (2.150) for $\tau \in [0, T_3]$. Let

$$F_3(\tau) := \sum_{j=2}^{k_0} |z_{j,3}(\tau)|^2 + 4|z_{1,3}(\tau)|^2. \tag{2.156}$$

To begin with,

$$\begin{aligned} z_{1,3}(\tau) &= z_{1,3}(0)\exp\left(\int_0^\tau (-1-\varepsilon_3(y) - r_3(y)F_3(y))\,dy\right) \\ &\quad + i h_0 \int_0^\tau \exp\left(\int_s^\tau (-1-\varepsilon_3(y) - r_3(y)F_3(y))\,dy\right) ds \\ &\quad - i \int_0^\tau \exp\left(\int_s^\tau (-1-\varepsilon_3(y) - r_3(y)F_3(y))\,dy\right) r_3(s) \sum_{n,j,l=2}^{k_0} \eta^l_{n,j}(u_{n,3}u_{j,3} + v_{n,3}v_{j,3}) z_{l,3}(s)\,ds. \end{aligned}$$

Using the explicit formula for ε_3, we calculate

$$\int_s^\tau \varepsilon_3(y)\,dy = \log\left(2\tau+\xi^2\right)^{1/2} - \log\left(2s+\xi^2\right)^{1/2}$$

and obtain

$$\begin{aligned} z_{1,3}(\tau) &= z_{1,3}(0)e^{-\tau}\frac{\xi}{(2\tau+\xi^2)^{1/2}}B(\tau,0) + \frac{i h_0}{(2\tau+\xi^2)^{1/2}}\int_0^\tau e^{s-\tau}(2s+\xi^2)^{1/2}B(\tau,s)\,ds \\ &\quad - \frac{i r_3(0)}{\xi(2\tau+\xi^2)^{1/2}}\int_0^\tau e^{s-\tau}(2s+\xi^2)\sum_{n,j,l=2}^{k_0}\eta^l_{n,j}B(\tau,s)(u_{n,3}u_{j,3}+v_{n,3}v_{j,3})z_{l,3}(s)\,ds, \end{aligned} \tag{2.157}$$

with

$$\begin{aligned} B(\tau,s) &:= \exp\left(-\int_s^\tau r_3(y)F_3(y)\,dy\right) \\ &= \exp\left(-\xi^{-1}r_3(0)\int_s^\tau (2y+\xi^2)^{1/2}F_3(y)\,dy\right). \end{aligned}$$

We employ once again a fixed point argument; letting

$$\mathcal{M}_1 = \left\{ z_{j,3} \in C([0,T_3]) : \sup_{\tau\in[0,T_3]} |z_{j,3}(\tau)| \le C_j, \text{ for } j = 2,\dots,k_0, \text{ with } \sum_{n,j=2}^{k_0} C_n^2 C_j \le \kappa \right\}$$

and considering $z_{j,3} \in \mathcal{M}_1, 2 \le j \le k_0$ in (2.157), we obtain that

$$\sup_{s\in[0,T_3]} |z^{a,r}_{1,3}(s)| \le C_1.$$

For $z_{k,3}$ we have

$$\begin{aligned} z_{k,3}(\tau) &= z_{k,3}(0)e^{-\tau}\frac{\xi}{(2\tau+\xi^2)^{1/2}}\exp\left(\frac{i b_k}{4a^2}\frac{\xi}{r_3(0)}(2\tau+\xi^2)^{1/2}\right) + \frac{i r_3(0)}{\xi(2\tau+\xi^2)^{1/2}}\int_0^\tau e^{s-\tau}(2s+\xi^2)\times \\ &\quad \times \exp\left(\frac{i b_k}{4a^2}\frac{\xi}{r_3(0)}\left[(2\tau+\xi^2)^{1/2}-(2s+\xi^2)^{1/2}\right]\right)\sum_{n,j,l=1}^{k_0}\theta^k_{n,j,l}(u_{n,3}u_{j,3}+v_{n,3}v_{j,3})z_{l,3}(s)\,ds \end{aligned} \tag{2.158}$$

and taking absolute values,

$$|z_{k,3}(\tau)| \le |z_{k,3}(0)| \frac{r_3(0)e^{-\tau}\xi}{(2\tau+\xi^2)^{1/2}} + \frac{10 r_3(0)}{\xi(2\tau+\xi^2)^{1/2}} \int_0^\tau e^{s-\tau}(2s+\xi^2)|z_{1,3}(s)|^2|z_{k,3}(s)|ds$$
$$+ e^{-\tau}\frac{r_3(0)}{\xi(2\tau+\xi^2)^{1/2}} \sum_{n,j=1,l=2}^{k_0} \theta^k_{n,j,l} \int_0^\tau e^s(2s+\xi^2)|z_{n,3}(s)||z_{j,3}(s)||z_{l,3}(s)|ds.$$

Using boundedness of $z_{1,3}$ and applying the Gronwall inequality yields

$$|z_{k,3}(\tau)| \le \tilde{C}|z_{k,3}(0)| \frac{e^{-\tau} r_3(0)\xi}{(2\tau+\xi^2)^{1/2}}$$
$$+ \tilde{C}\frac{e^{-\tau} r_3(0)}{\xi(2\tau+\xi^2)^{1/2}} \sum_{n,j=2,l=1}^{k_0} \theta^k_{n,j,l} \int_0^\tau e^s(2s+\xi^2)|z_{n,3}(s)||z_{j,3}(s)||z_{l,3}(s)|ds, \quad (2.159)$$

where $\tilde{C} = e^{10C_1^2}$. Taking the space

$$\mathscr{M} = \{z_k \in C([0,T_3]) : \sup_{\tau\in[0,T_3]} |z_{k,3}(\tau)| \le C_k r_3(0)\frac{e^{-s}\xi}{(2s+\xi^2)^{1/2}},$$
$$\text{for } 2 \le k \le k_0 \text{ with } \sum_{n,j=1}^{k_0} C_n^2 C_j \le \kappa\},$$

considering $\hat{z}_{k,3} \in \mathscr{M}$ in the sum in (2.158) instead of $z_{k,3}$ we obtain in the corresponding term in (2.159)

$$\int_0^\tau e^s(2s+\xi^2)|\hat{z}_{n,3}(s)||\hat{z}_{j,3}(s)||\hat{z}_{l,3}(s)|ds$$
$$\le C_n C_j C_l r_3(0)^2 \int_0^\tau e^{-s}\xi^2 ds \le C_n C_j C_l r_3(0)^2\xi^2.$$

Then applying in (2.159) the – by now – usual fixed-point argument, for sufficiently small $0 < r_3(0) < 1$ we obtain that $z_{k,3} \in \mathscr{M}$, i.e.

$$|z^{a,r}_{k,3}(\tau)| \le C_k r_3(0)\frac{e^{-\tau}\xi}{(2\tau+\xi^2)^{1/2}}.$$

Recall that our goal is to calculate the difference $w_j(\tau) := z^a_{j,3}(\tau) - z^r_{j,3}(\tau)$ at $\tau = T_3$ for $1 \le j \le k_0$. We begin with $w_1(\tau)$. The difference satisfies

$$z^a_{1,3}(\tau) - z^r_{1,3}(\tau) = (z^a_{1,3}(0) - z^r_{1,3}(0))\frac{e^{-\tau}\xi}{(2\tau+\xi^2)^{1/2}}$$
$$- \frac{i r_3(0)}{\xi(2\tau+\xi^2)^{1/2}} \int_0^\tau e^{s-\tau}(2s+\xi^2)\mathscr{B}^{a,r}(\tau,s)ds, \quad (2.160)$$

with

$$\mathscr{B}^{a,r}(\tau,s) = \sum_{n,j,l=1}^{k_0} \eta^l_{n,j}\Big(B^a(\tau,s)(u^a_{n,3}u^a_{j,3}+v^a_{n,3}v^a_{j,3})z^a_{l,3}(s) - B^r(\tau,s)(u^r_{n,3}u^a_{j,3}+v^r_{n,3}v^r_{j,3})z^r_{l,3}(s)\Big).$$

Consequently,

$$\begin{aligned}
|w_1(\tau)| &\le |w_1(0)|\frac{e^{-\tau}\xi}{(2\tau+\xi^2)^{1/2}} + \frac{r_3(0)}{\xi(2\tau+\xi^2)^{1/2}}\int_0^\tau e^{s-\tau}(2s+\xi^2)6(|z^a_{1,3}|^2+|z^r_{1,3}|^2)|w_1(s)|ds \\
&\quad + \frac{r_3(0)}{\xi(2\tau+\xi^2)^{1/2}}\int_0^\tau e^{s-\tau}(2s+\xi^2)\left[\sum_{k=2}^{k_0}\left(|z^a_{1,3}||z^a_{k,3}|^2+|z^r_{1,3}||z^r_{k,3}|^2\right)\right. \\
&\quad \left. + \sum_{n,j,l=2}^{k_0}\eta^l_{n,j}\left(|z^a_{n,3}||z^a_{j,3}||z^a_{l,3}|+|z^r_{n,3}||z^r_{j,3}||z^r_{l,3}|\right)\right] ds \\
&\le |w_1(0)|\frac{e^{-\tau}\xi}{(2\tau+\xi^2)^{1/2}} + \frac{r_3(0)}{\xi(2\tau+\xi^2)^{1/2}}\int_0^\tau e^{s-\tau}(2s+\xi^2)6(|z^a_{1,3}|^2+|z^r_{1,3}|^2)|w_1(s)|ds \\
&\quad + Ce^{-\tau}r_3(0)^3.
\end{aligned}$$

Observe that, for $0 \le \tau \le T_3$,

$$\begin{aligned}
\frac{r_3(0)}{\xi(2\tau+\xi^2)^{1/2}}\int_0^\tau e^{s-\tau}(2s+\xi^2)ds &\le \frac{r_3(0)(2\tau+\xi^2)}{\xi(2\tau+\xi^2)^{1/2}} \\
&= \frac{r_3(0)}{\xi}(2\tau+\xi^2)^{1/2} \le \frac{r_3(0)}{\xi}(2T_3+\xi^2)^{1/2} = 1.
\end{aligned}$$

Using the Gronwall inequality on $|w_1(\tau)|$ yields

$$|w_1(\tau)| \le e^{-\tau}\left(|w_1(0)|\frac{\xi}{(2\tau+\xi^2)^{1/2}} + Cr_3(0)^3\right)e^{12C_1^2} \tag{2.161}$$

and setting $\tau = T_3$,

$$\begin{aligned}
|w_1(T_3)| &\le \left(\frac{1}{\xi}h_0\sqrt{2\pi}e^{-\xi^2/2}e^{12C_1^2} + \mathcal{O}(r_3(0))\right)\frac{\xi}{\sqrt{2T_3+\xi^2}}e^{-T_3} + \mathcal{O}(r_3(0))e^{-T_3} \\
&= \left(h_0\sqrt{2\pi}e^{12C_1^2}\varepsilon^{1/2} + \mathcal{O}(\varepsilon)\right)e^{-\frac{1}{2\varepsilon}};
\end{aligned} \tag{2.162}$$

recall that $r_3(0) = \xi\varepsilon^{1/2}$.

For $w_k(T_3) = z^a_{k,3}(T_3) - z^r_{k,3}(T_3)$ we have

$$\begin{aligned}
z^a_{k,3}(\tau) - z^r_{k,3}(\tau) &= (z^a_{k,3}(0) - z^r_{k,3}(0))e^{-\tau}\frac{\xi}{(2\tau+\xi^2)^{1/2}}\exp\left(\frac{ib_k}{4a^2}\tau\right) \\
&\quad + \frac{ir_3(0)}{\xi(2\tau+\xi^2)^{1/2}}\int_0^\tau e^{s-\tau}(2s+\xi^2)\exp\left(\frac{ib_k}{4a^2}(\tau-s)\right)\Psi^{a,r}(s)ds,
\end{aligned} \tag{2.163}$$

where

$$\Psi^{a,r} = \sum_{n,j,l=1}^{k_0}\theta^k_{n,j,l}\Big((u^a_{n,3}u^a_{j,3}+v^a_{n,3}v^a_{j,3})z^a_{l,3}(s) - (u^r_{n,3}u^r_{j,3}+v^r_{n,3}v^r_{j,3})z^r_{l,3}(s)\Big).$$

Using estimates for $z^{a,r}_{k,3}, 2 \le k \le k_0$ from above and working as with $w_1(\tau)$ we find

$$|w_k(T_3)| = \mathcal{O}(\varepsilon)e^{-\frac{1}{2\varepsilon}}, \quad \text{for } \ 2 \le k \le k_0. \tag{2.164}$$

We have thus shown that

$$|w(\mu = 0)| = \left(h_0 C_d \varepsilon^{1/2} + \mathcal{O}(\varepsilon), \mathcal{O}(\varepsilon), \ldots, \mathcal{O}(\varepsilon)\right) e^{-\frac{1}{2\varepsilon}}, \tag{2.165}$$

where $C_d = \sqrt{2\pi} e^{12C_1^2}$, completing the proof of Proposition 2.31.

2.4 Concluding remarks

The primary contribution of this chapter is the replication of the result of [25] for the Galerkin discretisation of the corresponding PDE system with both fast and slow components being infinite-dimensional. It showcases the same method that was applied in the analysis of the fold singularity in the previous chapter and demonstrates that it is generic enough to be applied to a large variety of equations. Complementing the existence of slow-like manifolds shown in [30], our method provides a tool to handle situations where hyperbolicity is lost.

As in the previous chapter, loss of hyperbolicity in the PDE problem manifests as a $k_0 - 1$-dimensional submanifold separating normally hyperbolic parts of the critical manifold at the k_0-th truncation level. So far, we have been blowing up a single point of this submanifold. A blowup of the whole surface should be the next step as it may reveal aspects of the infinite-dimensional dynamics not yet explored and is an exciting direction for future research.

Most importantly, a more rigorous connection of the PDE and discretised dynamics would be desirable as it going to pave the way for a true extension of GSPT to infinite-dimensional problems.

Chapter 3

Bifurcation delay in a class of reaction-diffusion equations

3.1 Introduction

In this chapter, our goal is to compile a toolbox of geometric singular perturbation results that can be applied to fast-slow PDE-ODE system, where the fast variable is a PDE and the slow variables an ODE. The main motivation is to examine a more "geometric" way of recovering a set of results described in [15] concerning spatially homogeneous and non-homogeneous delayed onset of instabilities. The methods developed here allow us to only handle the spatially homogeneous case; despite this they are general enough to be applied to many PDE-ODE problems. As is common for geometric singular perturbation theory, our analysis is split into two parts, one for each the normally hyperbolic and non-hyperbolic regimes.

In the former, we provide a construction of locally invariant slow manifolds manifolds that is an analogue of Fenichel's theorem [21] for ODEs. The existence of slow manifold for such problems is not new; it is a special case of a much more general set of results found in [7, 8, 6, 9]. These works are however highly technical and accessible only to experts on infinite-dimensional manifold analysis and invariant manifold theory. Furthermore, the general theory has not been applied to fast-slow dynamics in an explicit way to the best of our knowledge. Thus, a self contained and accessible proof of existence of slow manifolds for fast-slow PDE-ODE systems would certainly be of interest. The proof provided here builds upon [5] where stable, unstable and center manifolds are constructed around steady states, and uses the same principles of the aforementioned work, namely obtaining a so called cone invariance result and using it to set up a graph transform, whose unique fixed point is the slow manifold.

For the non-hyperbolic regime, our approach is to use a form of the center manifold theorem, that appears in [24], and can be applied to infinite-dimensional, *i.e.* PDE problems and essentially allows the reduction to the corresponding ODE. The assumptions and the statement of the theorem are explained in much detail later in this chapter. This approach was independently employed in [3] where it is explored in depth and applied to many examples. However, the normally hyperbolic case is not considered in that work.

We begin by introducing the bifurcation delay results in [15] that motivated the work in this chapter.

3.1.1 Bifurcation delay in ODEs

An elementary example of an ODE exhibiting bifurcation delay is

$$\varepsilon \frac{\mathrm{d}u}{\mathrm{d}t} = a(t,\varepsilon)u, \tag{3.1}$$

where the function a is sufficiently smooth and for $\varepsilon = 0$ has a turning point at $t = t_*$, i.e.

$$a(t,0) < 0 \text{ for } t < t_*, \quad a(t,0) > 0 \text{ for } t > t_*.$$

The point t_* is called a *turning point*. By rewriting (3.1) as an autonomous fast-slow system

$$\varepsilon \frac{\mathrm{d}u}{\mathrm{d}t} = a(s,\varepsilon)u, \tag{3.2a}$$
$$\frac{\mathrm{d}s}{\mathrm{d}t} = 1, \tag{3.2b}$$

and switching to the fast time-scale τ using the rescaling $\tau = \frac{t}{\varepsilon}$,

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = a(s,\varepsilon)u, \tag{3.3a}$$
$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = \varepsilon, \tag{3.3b}$$

we enter the setting described in the previous paragraph. Observe that due to the turning point of a at $s = t_*$, for $\varepsilon = 0$, the first equation of the previous system undergoes a bifurcation at that value of s, where the steady state $u = 0$ loses its stability.

As a consequence of the simplicity of (3.1), the bifurcation delay can be analyzed using the explicit solution, given by

$$u(t) = u(0)\exp\left(\frac{1}{\varepsilon}\int_0^t a(s,\varepsilon)ds\right). \tag{3.4}$$

Consider the new quantity t_{exit}, defined uniquely by the relation

$$\int_0^{t_{\text{exit}}} a(s,0)ds = 0. \tag{3.5}$$

Due to the change of sign behaviour of $a(t,0)$, we have that $t_{\text{exit}} > t_*$. What is more, from (3.4), the solution is *exponentially small* in the interval $[t_*, t_{\text{exit}}]$ and t_{exit} is *independent* of ε.

By taking the limit $\varepsilon \to 0$, we find that

$$\lim_{\varepsilon\to 0} u(t;\varepsilon) = 0, \quad \text{for } t \in (0, t_{\text{exit}}),$$

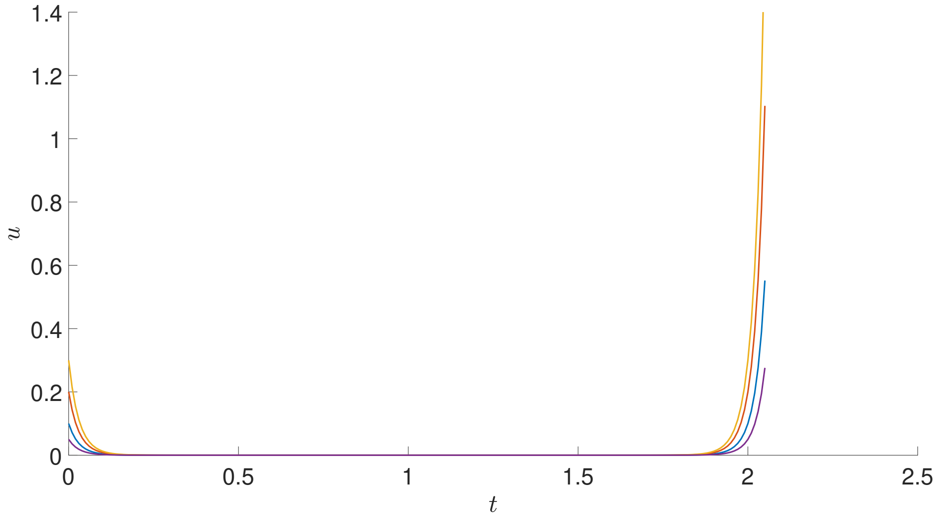


Figure 3.1: Plot of solutions to equation $\varepsilon \frac{du}{dt} = (t-1)u$, $\varepsilon = 0.03$ for various initial conditions. The bifurcation delay effect is evident, as the solutions stays close to the repelling half-line $\{t > 1\}$ for a large amount of time. Here, the turning point is at $t_* = 1$, while $t_{exit} = 2$.

$$\lim_{\varepsilon \to 0} u(t;\varepsilon) = \infty, \quad \text{for } t \in (t_{exit}, \infty).$$

Therefore we see that for small and positive ε, the solutions stay exponentially close to the steady state $u = 0$ after it becomes repelling for a strictly positive amount of time

$$t_{exit} - t_* = \mathcal{O}(1),$$

independent of ε (set Figure 3.1).

3.1.2 Bifurcation delay in a class of reaction-diffusion PDEs

Bifurcation delay and canard solutions have been observed in a class of singularly perturbed reaction-diffusion partial differential equations related to (3.1), namely equations of the form

$$\varepsilon u_t = \delta \varepsilon^{\alpha} u_{xx} + a(x,t,\varepsilon)u, \tag{3.6}$$

where $u(x,t)$ is defined in the spatial domain $x \in [0,1]$, $\delta > 0$ is a constant and $\alpha \geq 0$ with zero Neumann boundary conditions. For the C^2 function $a : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ we assume that there exists a $t_* \in C^1([0,1])$ curve of turning points $t_* = t_*(x)$, $t_*(x) > 0$, for $\varepsilon = 0$. In other words, $a(x, t_*(x), 0) = 0$ and

$$\begin{aligned} a(x,t,0) > 0 \quad & \text{for } t > t_*(x), \\ a(x,t,0) < 0 \quad & \text{for } t < t_*(x). \end{aligned}$$

Roughly speaking, the bifurcation delay in the context of this equation, is that the solution $u(x,t)$ will, for each $x \in [0,1]$, remain exponentially small for an $\mathcal{O}(1)$ amount of time after the corresponding turning point $t_*(x)$ is crossed, before switching to an exponentially increasing behaviour. The onset of this increasing behaviour is called exit time and a priori we expect it to vary with x. The precise meaning of the exit time in case of (3.6) is analogous to that of (3.1).

Definition 3.1. The exit time $t_{\text{exit}}(x)$ for $x \in [0,1]$ is the (positive) real number so that a solution $u(x,t)$ of (3.6) satisfies

$$\lim_{\varepsilon\to 0} u(x,t;\varepsilon) = 0, \quad \text{for } t \in (0, t_{\text{exit}}(x)),$$
$$\lim_{\varepsilon\to 0} u(x,t;\varepsilon) = \infty, \quad \text{for } t \in (t_{\text{exit}}(x), \infty).$$

In case the function $t_{\text{exit}}(x)$ is constant in x, we will write t_{PDEexit} for it and say that the exit time is homogeneous. Otherwise, the exit time is non-homogeneous.

In this context, we say that (3.6) exhibits bifurcation delay if

$$\{x \in [0,1] \mid t_{\text{exit}}(x) > t_*(x)\} \neq \emptyset.$$

The first result in this direction, for $\alpha = 1$, was obtained in [43], see also [13] for an exposition of the topic by the same authors. There, it is proved that the bifurcation delay exists using a method that involves finding upper and lower solutions to (3.6). The results where extended for general $\alpha \geq 0$ in [15] using a similar method, while expressions for the exit time where also given.

Theorem 3.2 ([15, Theorem 2])**.** *Under the previous assumptions, and assuming that the initial condition $u(x,0)$ is strictly positive in $[0,1]$, equation (3.6) satisfies:*

- *for $\alpha < 2$ the solution has a homogeneous exit time determined by*

$$\int_0^{t_{\text{PDEexit}}} \max_x a(x,t,0)\,dt = 0.$$

- *for $\alpha > 2$ the solution has a non-homogeneous exit time $t_{\text{exit}}(x)$ determined by*

$$\int_0^{t_{\text{exit}}(x)} a(x,t,0)\,dt = 0.$$

At the critical value of the exponent $\alpha = 2$, it is expected that the behaviour varies with δ. Numerical examples for the particular choices $\alpha = 0$ and $\alpha = 3$ can be seen in Figures 3.2, 3.3, 3.4.

3.1.3 Scope of this chapter

The aim of this project is to investigate whether a more "geometric" (qualitative) analysis the problem is possible. We examine only the simplest case $\alpha = 0$. This may seem like an unreasonable simplification and in a way it is; however recall that our goal – inspired from this spatio-temporal bifurcation delay problem – is mainly to develop GSPT for PDE-ODE systems. The strategy used in the analysis is described below:

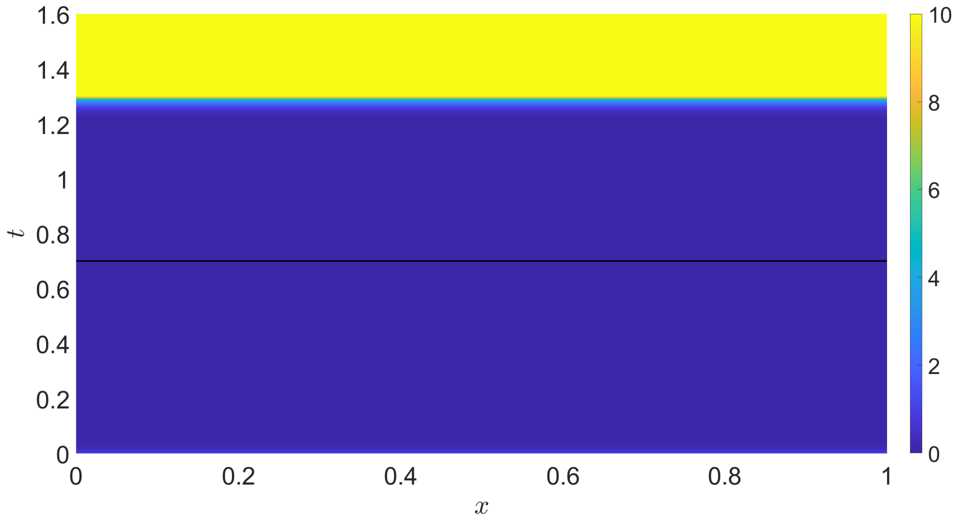


Figure 3.2: Numerical solution of $\varepsilon u_t = u_{xx} + (t - 0.7)u$, with $\varepsilon = 10^{-3}$, illustrating the bifurcation delay. The black solid line is the curve of turning points, here given by $t = 0.7$. Observe that, while the stability is lost at $t_* = 0.7$, the solution stays exponentially close to 0 for a large amount of time, before switching to increasing behaviour at about $t_{\text{PDEexit}} = 1.4$, as predicted the homogeneous exit time of Theorem 3.2. The initial condition is $u(x,0) = 0.5$. As after the exit time, the solution increases like $\exp(1/\varepsilon)$, all the values larger that 10 are colored yellow.

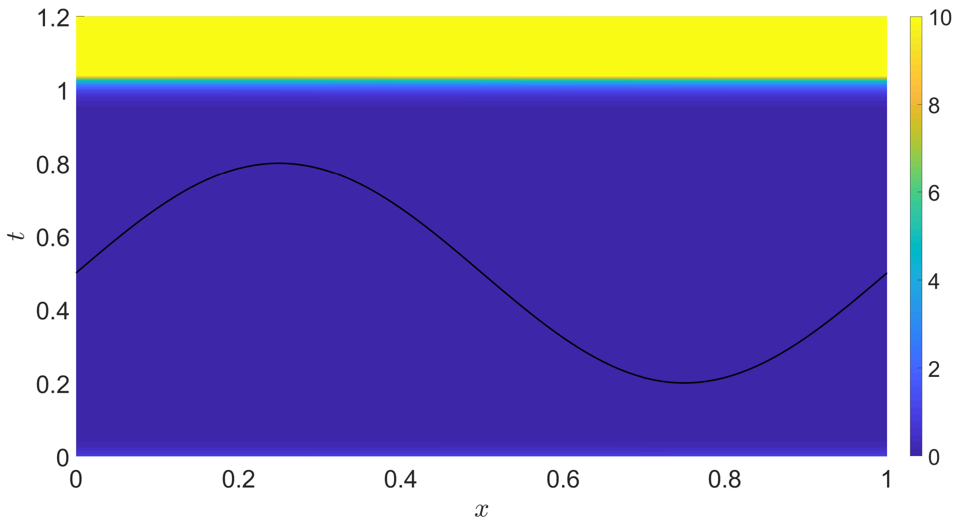


Figure 3.3: Numerical solution of $\varepsilon u_t = u_{xx} + \left(t - \frac{1}{2} - \frac{3}{10}\sin(2\pi x)\right)u$ for $\varepsilon = 10^{-3}$. In this case, the turning point depends on x and is given by $t_*(x) = \frac{1}{2} - \frac{3}{10}\sin 2\pi x$. However the exit time is still homogeneous in space, as asserted in Theorem 3.2. Intuitively, the diffusion is fast enough to homogenize the solution. The initial condition is again $u(x,0) = 0.5$.

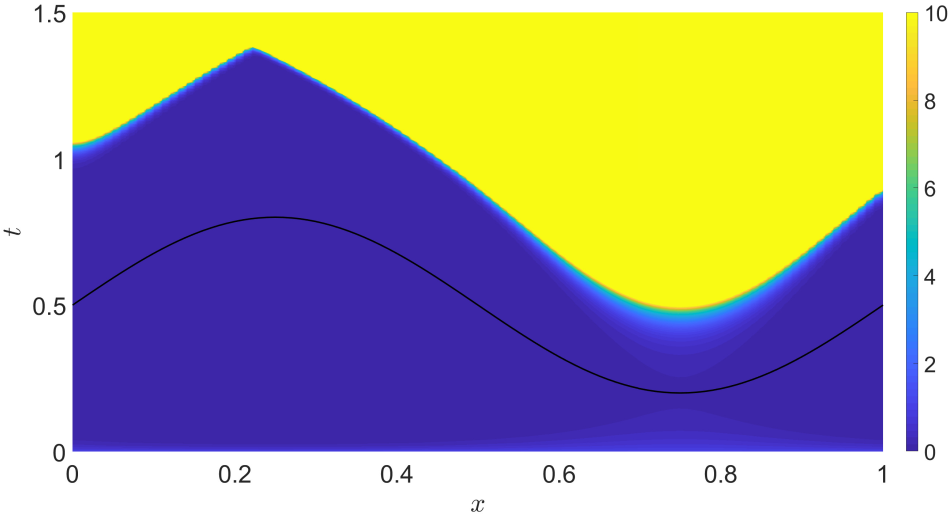


Figure 3.4: Numerical solution of $\varepsilon u_t = \varepsilon^3 u_{xx} + \left(t - \frac{1}{2} - \frac{3}{10}\sin(2\pi x)\right) u$ for $\varepsilon = 10^{-3}$. The turning point is given by $t_*(x) = \frac{1}{2} - \frac{3}{10}\sin 2\pi x$. This time though, the exit time is not homogeneous in space, as we are in the second case of Theorem 3.2. Intuitively, the diffusion is not fast enough to homogenize the solution. The initial condition is $u(x,0) = 0.5$.

- Write (3.6) as an autonomous system and rescale time by $t \mapsto \frac{t}{\varepsilon}$:

$$\partial_t u = \partial_x^2 u + a(x,s,\varepsilon)u, \tag{3.7a}$$
$$\partial_t s = \varepsilon. \tag{3.7b}$$

  Recall that the first equation is accompanied by zero Neumann boundary conditions.

- Define the "critical manifold" $\mathscr{C}_0 := \{(u,s) : u_{xx} + a(x,s,0)u = 0, s < s_*\}$, where s_* it the smallest value of s such that the above linear operator has a zero eigenvalue; see subsection 3.4.3. This set is a curve in $L^2([0,1])$ and is normally hyperbolic attracting in the sense described in the following section and thus perturbs to an attracting slow manifold.

- At $s = s_*$, normal hyperbolicity is lost and we use the center manifold theorem. Define the differential operator

$$L_{s,\varepsilon} u = u_{xx} + a(x,t,\varepsilon)u,$$

  so that $\partial_t u = L_{s,\varepsilon} u$ and apply the *center manifold theorem* to analyze the system around this critical value, by reducing it to an ODE system.

- Observe the bifurcation delay behaviour in the reduced ODE, which turns out to be of the form (3.3).

The rest of this chapter is structured as follows: in section 3.2 we construct slow manifolds PDE-ODE systems; in section 3.3 we introduce the center manifold theorem and in section 3.4 we to apply it to (3.7).

3.2 Slow manifolds for fast-slow PDE-ODE systems

3.2.1 Setup

We start with the PDE-ODE system

$$u_t = \Delta u + f(u,v,\varepsilon), \qquad u_x(\pm\ell, t) = 0 \tag{3.8a}$$
$$v_t = \varepsilon g(u,v,\varepsilon) \tag{3.8b}$$

with $u(x,t) \in \mathbb{R}, x \in [-\ell,\ell]$, zero Neumann boundary conditions and $f, g : \mathbb{R}^3 \to \mathbb{R}$ are smooth functions. We consider a critical manifold $\mathscr{C}_0$ consisting of constant in space functions, so that

$$\mathscr{C}_0 := \{(u,v) \in \mathbb{R}^2 : f(u,v,0) = 0\}. \tag{3.9}$$

The "normal hyperbolicity" condition we assume is that for any $(u_0, v_0) \in \mathscr{C}_0$ it holds that $\frac{\partial}{\partial u} f(u_0, v_0, 0) < 0$. We also assume that $\mathscr{C}_0$ is compact so all these negative derivatives are away from zero. We assume that $\mathscr{C}_0$ in normally hyperbolic so it can be written as a graph

$$\mathscr{C}_0 = \{u = G(v) : v \in [a,b]\} \tag{3.10}$$

where $G : \mathbb{R} \to \mathbb{R}$ is smooth. Without loss of generality, we may take $G \equiv 0$, by considering $\tilde{u} = u - f(v)$ in (3.8).

Remark 3.3. We can also consider more general boundary conditions and functions f that have a spatial dependence in (3.8), that is, $f = f(x,u,v,\varepsilon)$. Then, $\mathscr{C}_0$ will consist of functions in $H^2([-\ell,\ell])$ such that

$$\mathscr{C}_0 = \{u \in H^2([-\ell,\ell]) : \Delta u + f(x,u,v,0) = 0\}.$$

Then $\mathscr{C}_0$ will still be a curve parametrised by $v \in \mathbb{R}$ and the hyperbolicity condition can be replaced by demanding that the semigroup generated by the Fréchet derivative $D_u\big(\Delta u + f(x,u,v,\varepsilon)\big)$ satisfies an exponential decay property, such as (3.14). For clarity of presentation, we work with spatially homogeneous solutions.

As is usual when working with center-like manifolds we have to modify the relevant directions to show persistence results. In our case, we have to modify the slow equation so that $\mathscr{C}_0$ is "overflowing", in the sense that the v-component of the vector field points outwards at the boundary of $\mathscr{C}_0$ for sufficiently small $\varepsilon > 0$. Choose a small $\rho > 0$ and consider a function $j : \mathbb{R} \to \mathbb{R}$ with the following properties:

- $j \in C^\infty(\mathbb{R},\mathbb{R})$ increasing,
- $j(v) = 0$ for $v \in [a,b]$,
- $j(v) = -1$ for $v \in (-\infty, a-\rho]$,
- $j(v) = 1$ for $v \in [b+\rho, \infty)$.

Such a function exists from standard analysis results. In place of (3.8) we work with

$$u_t = \Delta u + f(u,v,\varepsilon), \tag{3.11a}$$
$$v_t = \varepsilon g(u,v,\varepsilon) + j(v). \tag{3.11b}$$

Note that the new vector field coincides with the unmodified one for $v \in [a, b]$. Throughout the following we will be working in a neighbourhood of $\mathscr{C}_0$

$$\mathscr{U} := \{(u, v) \in L^2([-\ell, \ell]) \times \mathbb{R} : |u| \le C_u, v \in [a - \rho, b + \rho]\} \tag{3.12}$$

and consider $\varepsilon \in [0, \varepsilon_0]$, where $C_u, \varepsilon_0 > 0$ are fixed, small, positive constants to be determined later.

Define $X = L^2([-\ell, \ell]) \times \mathbb{R}$. Then $X = X^s \oplus X^c$, where $X^s = L^2([-\ell, \ell]) \times \{0\}$ and $X^c = \{0\} \times \mathbb{R}$. However we shall identify X with $X^c \times X^s$, where misusing notations we have $X^s = L^2([-\ell, \ell])$ and $X^c = \mathbb{R}$. Problem (3.11) generates a semiflow Φ_t on X.

The first restriction we impose of ε_0 is that

$$\varepsilon g(u, a - \rho, \varepsilon) + j(a - \rho) \le -\frac{1}{2},$$
$$\varepsilon g(u, b + \rho, \varepsilon) + j(b + \rho) \ge \frac{1}{2}$$

holds on $\mathscr{U}$ for all $0 \le \varepsilon \le \varepsilon_0$. Furthermore, by restricting C_u, we may assume that f has Lipschitz constant $\eta > 0$. We will need the following version of Gronwall's inequality:

Lemma 3.4. *Let* $f : [0, T] \to [0, \infty)$ *be continuous. If*

$$f(t - \tau) \le f(t) + c \int_{-\tau}^{0} f(t + s) ds$$

for $-T \le -t \le -\tau \le 0$, *then*

$$f(t) \ge f(0) e^{-ct}, \quad 0 \le t \le T.$$

Proof. This is [5][Lemma 2.2]. Apply Gronwall's inequality to $f(t - \tau)$ for fixed $t > 0$ with $\tau \in [0, t]$ as the variable and set $\tau = t$. □

3.2.2 Cone invariance

The key ingredient to constructing a slow manifold is the following statement on invariant cones for the semiflow of (3.8) illustrated in Figure 3.5. The statement that follows holds for any $\mu > 0$, but we restrict our attention to $\mu \in [1 - \zeta, 1 + \zeta]$ were $0 < \zeta < 1/2$ is a constant.

Proposition 3.5. *Let* $(u_2(0), v_2(0)), (u_1(0), v_1(0)) \in \mathscr{U}$ *with* $v_1(0), v_2(0) \in [a, b]$. *If*

$$|u_2(0) - u_1(0)| \le \mu |v_2(0) - v_1(0)|$$

then

$$|u_2(t) - u_1(t)| \le \lambda \mu |v_2(t) - v_1(t)|$$

for some $0 < \lambda < 1$ *and sufficiently small* $t > 0$ *such that* $v_1(t), v_2(t) \in [a, b]$.

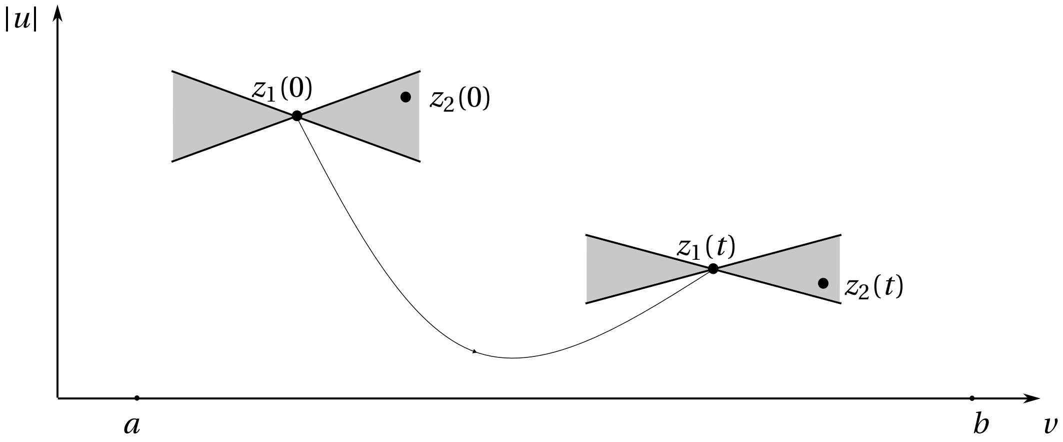


Figure 3.5: Illustration of the cone invariance: if $z_2(0)$ is in the cone of $z_1(0)$ and we evolve both points forward for time $t > 0$, $z_2(t)$ will be in a tighter cone of $z_1(t)$, provided we remain over the critical manifold.

Proof. Let $\omega := \frac{\partial}{\partial u} f(0, v_1(t), 0) < 0$. If we expand f around $(0, v_1, 0)$ we obtain

$$\begin{aligned} f(u, v, \varepsilon) &= f(0, v_1, 0) + \omega u + F(u, v, \varepsilon) + \mathcal{O}(\varepsilon) \\ &= \omega u + F(u, v, \varepsilon), \end{aligned}$$

because $(0, v_1) \in \mathcal{C}_0$, where $F(u, v, \varepsilon) = \mathcal{O}(u^2, v, \varepsilon)$. Then (3.8) becomes

$$u_t = (\Delta + \omega) u + F(u, v, \varepsilon), \tag{3.13a}$$

$$v_t = \varepsilon g(u, v, \varepsilon). \tag{3.13b}$$

The semigroup $S_1(t)$ over $L^2([-\ell, \ell])$ generated by the operator $\Delta + \omega$ satisfies

$$\|S_1(t)\| \leq C_\Delta e^{\omega t}. \tag{3.14}$$

To obtain more favourable estimates, we renorm $(L^2([-\ell, \ell]), |\cdot|)$ with the equivalent norm $|\cdot|_1$ given by

$$|u|_1 = \sup_{t \geq 0} e^{-\omega t} |S_1(t) u|, \quad u \in L^2([-\ell, \ell]).$$

Indeed, $|u| \leq |u|_1 \leq C_\Delta |u|$ so that the two norms are equivalent. In the operator norm $\|\cdot\|_1$ derived from $|\cdot|_1$, in place of (3.14) we have

$$\|S_1(t)\|_1 \leq e^{\omega t},$$

that is, we replaced C_Δ with 1; see [5] for further discussion on the renorming process. From this point on and for the rest of the section, we replace $|\cdot|$ with $|\cdot|_1$ while keeping the former notation.

The variation of constants formula gives

$$u_i(t) = S_1(t) u_i(0) + \int_0^t S_1(t-s) F(u_i(s), v_i(s), \varepsilon) ds \tag{3.15}$$

for $i = 1,2$ and

$$u_1(t)-u_2(t) = S_1(t)(u_1(0)-u_2(0))+\int_0^t S_1(t-s)(F(u_1(s),v_1(s),\varepsilon)-F(u_2(s),v_2(s),\varepsilon))ds \tag{3.16}$$

Taking norms,

$$|u_1(t)-u_2(t)| \le e^{\omega t}|u_1(0)-u_2(0)|+\eta e^{\omega t}\int_0^t e^{-\omega s}(|u_1(s)-u_2(s)|+|v_1(s)-v_2(s)|)ds. \tag{3.17}$$

where we used that F is Lipschitz with constant η.

Recall that by assumption, $|u_2(0)-u_1(0)| \le \mu|v_2(0)-v_1(0)|$ and take ξ such that $0<\xi<\zeta$. From this, and the the continuity of the semiflow, for small times $t \in [0, t_\xi]$, the values of u_1, u_2, v_1, v_2 will not change much and it will hold that

$$(\mu-\xi)|v_2(t)-v_1(t)| \le |u_2(t)-u_1(t)| \le (\mu+\xi)|v_2(t)-v_1(t)| \tag{3.18}$$

Using this in (3.17), we obtain

$$|u_2(t)-u_1(t)| \le |u_2(0)-u_1(0)|\exp\left[(\omega+\eta(1+(\mu-\xi)^{-1}))t\right] \tag{3.19}$$

for $t \in [0, t_\xi]$ by application of the Gronwall inequality.

Similarly for the difference $|v_2(t)-v_1(t)|$, we integrate the v equation from $t+\tau$ to t where $-t \le \tau \le 0$ we find

$$|v_2(t+\tau)-v_1(t+\tau)| \le |v_2(t)-v_1(t)|+\varepsilon\theta\int_\tau^0 (|u_2(t+s)-u_1(t+s)|+|v_2(t+s)-v_1(t+s)|)ds \tag{3.20}$$

with θ the Lipschitz constant of g and using (3.18) with Lemma 3.4,

$$|v_2(t)-v_1(t)| \ge |v_2(0)-v_1(0)|\exp\left[-\varepsilon\theta(1+\mu+\xi)t\right]. \tag{3.21}$$

Thus,

$$\begin{aligned}\frac{|u_2(t)-u_1(t)|}{|v_2(t)-v_1(t)|} &\le \frac{|u_2(0)-u_1(0)|}{|v_2(0)-v_1(0)|}\exp\left[\left(\omega+\eta(1+(\mu-\xi)^{-1})+\varepsilon\theta(1+\mu+\xi)\right)t\right] \\ &\le \mu\exp\left[\left(\omega+\eta(1+(\mu-\xi)^{-1})+\varepsilon\theta(1+\mu+\xi)\right)t\right].\end{aligned}$$

To prove the result it would suffice

$$\omega+\eta(1+(\mu-\xi)^{-1})+\varepsilon\theta(1+\mu+\xi) < 0, \tag{3.22}$$

where θ is the Lipschitz constant of g, which can be achieved by by taking ε_0, η small enough so that $\omega+\eta(2+\zeta)+\varepsilon\theta(2+\zeta) < 0$. □

Remark 3.6. The constant ω can be chosen independently of $(0, v_1)$ to be

$$\omega = \sup_{v\in[a,b]} \frac{\partial}{\partial u} f(0,v,0) < 0$$

as the interval $[a,b]$ is compact. In addition the constant C_Δ in (3.14) depends on the space domain only; thus the renorming is also independent of any linearisation we consider in the proof.

Remark 3.7. The slow manifold we will construct is only locally invariant in the sense that orbits will leave the neighbourhood $\mathscr{U}$ along the center v-direction due to the overflowing modification. Hence, our analysis holds only as long as we remain in the interval $[a,b]$.

Define the cone with opening μ by

$$K_\mu := \{(u,v) : |u| \le \mu|v|\} \cap \mathscr{U}. \tag{3.23}$$

Another way to state the cone invariance is that

$$\text{if}\,(u_2(0), v_2(0)) \in (u_1(0), v_1(0)) + K_\mu \text{ then } (u_2(t), v_2(t)) \in (u_1(t), v_1(t)) + K_\mu \tag{3.24}$$

for all $t > 0$ such that $v_1(t), v_2(t) \in [a,b]$. This follows by applying Proposition 3.5 – which holds for small times – repeatedly.

3.2.3 Construction of slow manifolds

The locally invariant slow manifold will be constructed as the graph of a function in the space of Lipschitz functions with Lipschitz constant less that μ,

$$\mathscr{V} = \mathscr{V}(\mu) := \{h \in C([a,b], L^2(-\ell,\ell)) : |h(v_1) - h(v_2)| \le \mu|v_1 - v_2|\}, \tag{3.25}$$

endowed with the supremum norm

$$\|h\| = \sup_{v\in[a,b]} |h(v)|.$$

It is known that $(\mathscr{V}(\mu), \|\cdot\|)$ is a complete normed space.

Remark 3.8. It is important that all the functions $h \in \mathscr{V}(\mu)$ have $\mathrm{Lip}(h) \le \mu$. If we consider just Lipschitz functions with the same norm, we do not have the completeness required to apply the fixed point theorem. As we shall see, the cone invariance implies that the graph transform does not increase the Lipschitz constant of the function it is applied to.

Let $z := (u,v) \in L^2([-\ell,\ell]) \times \mathbb{R}$ and recall that $\Phi_t(z)$ denotes the semiflow of the problem. Let $\Pi : X \to \mathbb{R}$ be the projection $\Pi z = v$, For a function $h \in \mathscr{V}$, define H as the graph of h, that is

$$H = \{(h(v), v) : v \in [a,b]\}.$$

Lemma 3.9. *If* $h \in \mathscr{V}$ *then* $\Pi\Phi_t(H) \supseteq [a,b]$ *for any* $t > 0$.

Proof. We use a topological argument, invoking the Brouwer degree invariance. For $h \in \mathscr{V}$, consider the extension $\hat{h} : [a-\rho, b+\rho] \to L^2([-\ell,\ell])$ given by

$$\hat{h}(v) = \begin{cases} h(a), & \text{if } v \in [a-\rho, a], \\ h(v), & \text{if } v \in [a,b], \\ h(b), & \text{if } v \in [b, +\rho], \end{cases} \tag{3.26}$$

which still is a μ-Lipschitz function. Define the family of maps

$$\phi_s(v) = \Pi\Phi_s(\hat{h}(v), v)$$

for $s \in [0, t]$, $v \in [a-\rho, b+\rho]$. Due to the semiflow property, ϕ is jointly continuous on $(s, v) \in [0, t] \times [a-\rho, b+\rho]$ and for $s = 0$ it reduces to the identity map. For $v = a-\rho, b+\rho$, because of the overflowing modification made to the v-equation, we have that $\phi_s(a-\rho) < a-\rho$ and $\phi_s(b+\rho) > b+\rho$ for any $s \in [0, t]$. By homotopy invariance of the Brouwer degree,

$$\mathrm{d}\left(\phi_s(\cdot), [a, b], v_0\right) = \mathrm{d}\left(\phi_0(\cdot), [a, b], v_0\right) = 1$$

for any $v_0 \in [a-\rho, b+\rho]$ and the proof is complete. □

Lemma 3.10. *$\Phi_t(H)$ is the graph of a function in $\mathcal{V}$ for all $t \geq 0$.*

Proof. This is a straightforwards consequence of the cone invariance. If $z_1, z_2 \in \Phi_t(H)$ then there exist $z_1^0, z_2^0 \in H$ such that $z_1 = \Phi_t\left(z_1^0\right), z_2 = \Phi_t\left(z_2^0\right)$. Since h is Lipschitz with constant μ, $z_2^0 \in z_1^0 + K_\mu$. By Proposition 3.5, $z_2 \in z_1 + K_\mu$, thus $\Phi_t(H)$ is the graph of an μ-Lipschitz function. □

The last two statements define a map $\mathcal{G}_t : \mathcal{V} \to \mathcal{V}$ for each $t \geq 0$, commonly referred to as the graph transform.

Lemma 3.11. *The map $\mathcal{G}_t : \mathcal{V} \to \mathcal{V}$ is a contraction for sufficiently large $t > 0$.*

Proof. Let $h_1, h_2 \in \mathcal{V}$ and $\tilde{h}_1 = \mathcal{G}_t(h_1), \tilde{h}_2 = \mathcal{G}_t(h_2)$ and take any $v_0, \in [a, b]$. Then, there exist $v_1, v_2 \in [a, b]$ such that

$$\begin{aligned}(\tilde{h}_1(v_0), v_0) &= \Phi_t(h_1(v_1), v_1)\\ (\tilde{h}_2(v_0), v_0) &= \Phi_t(h_2(v_2), v_2).\end{aligned}$$

Let $\beta > 1 + \zeta > \mu$. Since

$$\left(\tilde{h}_1(v_0), v_0\right) \notin \left(\tilde{h}_2(v_0), v_0\right) + K_\beta,$$

unless $\tilde{h}_1(v_0) = \tilde{h}_2(v_0)$, as a consequence of the cone invariance,

$$\Phi_s\left(h_1(v_1), v_1\right) \notin \Phi_s\left(h_2(v_2), v_2\right) + K_\beta, \tag{3.27}$$

for $0 \leq s \leq t$. In particular,

$$|h_1(v_1) - h_2(v_2)| \geq \beta |v_1 - v_2|. \tag{3.28}$$

Using (3.27) in (3.17) we find

$$|\tilde{h}_1(v_0) - \tilde{h}_2(v_0)| \leq |h_1(v_1) - h_2(v_2)| \exp\left[(\omega + \eta(1+\beta^{-1}))t\right]. \tag{3.29}$$

Also,

$$\begin{aligned}|h_1(v_1) - h_2(v_2)| &\le |h_1(v_1) - h_2(v_1)| + |h_2(v_1) - h_2(v_2)| \\ &\le |h_1(v_1) - h_2(v_1)| + \mu|v_1 - v_2| \\ &\le |h_1(v_1) - h_2(v_1)| + \frac{\mu}{\beta}|h_1(v_1) - h_2(v_2)|\end{aligned}$$

using (3.28) and rearranging,

$$|h_1(v_1) - h_2(v_2)| \le \frac{\beta}{\beta-\mu}|h_1(v_1) - h_2(v_1)|. \tag{3.30}$$

Replacing into (3.29),

$$|\tilde{h}_1(v_0) - \tilde{h}_2(v_0)| \le \frac{\beta}{\beta-\mu}|h_1(v_1) - h_2(v_1)| \exp\left[(\omega + \eta(1+\beta^{-1}))t\right] \tag{3.31}$$

and taking supremum over $v_0 \in [a, b]$,

$$\|\tilde{h}_1 - \tilde{h}_2\| \le \frac{\beta}{\beta-\mu}\|h_1 - h_2\| \exp\left[(\omega + \eta(1+\beta^{-1}))t\right]. \tag{3.32}$$

Since $\omega + \eta\left(1+\beta^{-1}\right) < 0$, the graph transform is a contraction on $\mathscr{V}$ for large $t > 0$. □

We deduce that $\mathscr{G}_t$ has a unique fixed point $\tilde{h}_t$ for each sufficiently large t. By definition of $\mathscr{G}_t$, the graph $\tilde{H}_t$ of $\tilde{h}_t$ is locally invariant under the semiflow Φ_t.

Lemma 3.12. *The function $\tilde{h}_t$ is independent of t*

Proof. Let $\tau \ge t > 0$ with t sufficiently large so that $\mathscr{G}_t$ is a contraction. By definition, $\mathscr{G}_\tau, \mathscr{G}_t$ commute, so that

$$\mathscr{G}_t\mathscr{G}_\tau(\tilde{h}_t) = \mathscr{G}_\tau\mathscr{G}_t(\tilde{h}_t) = \mathscr{G}_\tau\tilde{h}_t,$$

which means that $\mathscr{G}_\tau\tilde{h}_t$ is a fixed point of $\mathscr{G}_t$. By the uniqueness of the fixed point of $\mathscr{G}_t$ it follows that $\mathscr{G}_\tau\tilde{h}_t = \tilde{h}_t$, which in turn implies that $\tilde{h}_t$ is the fixed point of $\mathscr{G}_\tau$ for all τ. □

Therefore, we denote $\tilde{h}_t$ by $\tilde{h}$ and let $\tilde{H} = \operatorname{graph}\tilde{h}$.

Lemma 3.13. *There exists $c < 0$ such that for any $z_0 = (u_0, v_0) \in \mathscr{U}$ with $v_0 \in [a, b]$, we have* $\operatorname{dist}(z_0(t), \tilde{H}) \le |u_0 - \tilde{h}(v_0)|e^{ct}$ *for all $t > 0$ such that $v_0(t) \in [a, b]$.*

Proof. We estimate the difference between the points z_0 and $z_1 = (\tilde{h}(v_0), v_0)$. Because z_0, z_1 have the same v coordinate, we have that

$$z_0 \notin z_1 + K_\mu.$$

By the cone invariance property,

$$\Phi_t z_0 \notin \Phi_t z_1 + K_\mu,$$

for any t such that $\Pi\Phi_t z_0, \Pi\Phi_t z_1 \in [a,b]$. Using this property in (3.17) for z_0, z_1,

$$|u_0(t) - \tilde{h}(v_0)(t)| \le |u_0 - \tilde{h}(v_0)| \exp\left[(\omega + \eta(1+\mu^{-1}))t\right],$$

and the lemma is proved with $c = \omega + \eta(1+\mu^{-1}) < 0$. □

We have established the existence of exponentially attracting locally invariant slow manifolds for a class of fast-slow ODE-PDE systems under the notion of normal hyperbolicity described earlier. The manifold constructed will in general not be unique, despite the graph transform having a unique fixed point. This is deceiving, as different overflowing modifications and cutoffs of the original problem make the slow manifold non-unique, something that is standard in such constructions involving center-like dynamics. The smoothness properties of $\tilde{h}$ should be discussed. Our $\tilde{h}$ is only Lipschitz continuous, but it can be shown to be of C^k class for any $k > 0$ but not C^∞ [9] – not something unexpected in such constructions.

The problem we are studying in the subsequent sections is of the form

$$u_t = \delta u_{xx} + f(u,x,s,\varepsilon), \tag{3.33a}$$

$$s_t = \varepsilon, \tag{3.33b}$$

for a constant $\delta > 0$ and for different forms of the function f with a turning curve property as described in the introduction. This type of problem is covered by the slow manifold existence result we just gave and has a critical manifold given by

$$\mathscr{C}_0 = \{(u,s) : \delta u_{xx} + f(u,x,s,0) = 0, s \in [-C, s_*]\},$$

where $C > 0$ is an arbitrary constant. This set of steady states of the fast equation for $\varepsilon = 0$ is normally hyperbolic up to some point s_*; see section 3.4 and the system has a slow manifold for $\varepsilon > 0$ small. At $s = s_*$, hyperbolicity is lost and we use the center manifold theorem, given below.

3.3 The center manifold theorem

In this section, we introduce the *center manifold theorem in infinite dimensions*, the main tool used for the analysis of the bifurcation delay occurring in the linear partial differential equation

$$\varepsilon u_t = \delta u_{xx} + a(x,t,\varepsilon)u$$

and its non-linear version

$$\varepsilon u_t = \delta u_{xx} + a(x,t,\varepsilon)u + f(u,x,t,\varepsilon).$$

Remark 3.14. This form of the center manifold theorem, as will become evident, in the context of fast-slow systems can only be applied when *the slow variable is finite-dimensional*, thus rendering it inadequate to treat PDE-PDE systems such as those in chapter 1 and chapter 2.

The main reference for this section is [24, Chapter 2], where most of the results presented here come from. The interested reader is encouraged to consult the aforementioned book for a wealth of information regarding the history of the theorem, as well as numerous examples of applications to a variety of problems. Here, we restrict ourselves to stating the versions of the theorem which are the most relevant for our purposes.

The center manifold theorem applies to differential equations of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = F(u),$$

where u is an X-valued function of t, where X is an arbitrary Banach space, possibly infinite dimensional and $F(u) = 0$, so that the solution $u = 0$ is a steady state of the equation. The first step, as in the case of a ODE system, is to linearise the equation and write:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Lu + R(u),$$

where $Lu = DF(0)$ is the Fréchet derivative of F, a linear operator, and $R(u)$ satisfies $DR(0) = 0$. All the derivatives appearing from now on should be understood in the Frechét sense. For a short introduction to calculus on Banach spaces, see [2]. This formulation is fairly general, as for example, when L is a differential operator, the last equation corresponds to a parabolic partial differential equation. An other class of equations that may be brought to this form are delay differential equations (DDEs).

The next step is to examine the spectrum of the operator L. In case that spectrum on the imaginary axis, the *center spectrum*, consists of a finite number of eigenvalues of finite algebraic multiplicity only, under certain assumptions on the rest of the spectrum and on properties of L, it may possible to project the problem onto a finite dimensional subspace, thus reducing a possibly infinite-dimensional problem to a *finite-dimensional* one.

3.3.1 Definitions and assumptions

As already mentioned, we examine equations of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Lu + R(u). \tag{3.34}$$

To be precise, consider three Banach spaces, real or complex, X, Y, Z such that $Z \hookrightarrow Y \hookrightarrow X$, with continuous embeddings.

Definition 3.15 ([24])**.** Let $C^k(Z, X)$ denote the space of k-times (Frechét) differentiable functions $F : Z \to X$, with norm

$$\|F\|_{C^k} = \max_{j=0,\dots,k} \sup_{y \in Z} \|D^j F(y)\|_{\mathscr{L}(Z^j, X)}.$$

As usual, $\mathscr{L}(Z, X)$ denotes the space of (bounded) linear operators $Z \to X$ with the operator norm. Also, the following two spaces will appear in the statement of the theorem:

- $C_\eta(\mathbb{R}, X) := \left\{u \in C^0(\mathbb{R}, X) \text{ with } \|u\|_{C_\eta} = \sup_{t\in\mathbb{R}} e^{-\eta|t|}\|u(t)\|_X < \infty\right\}$, the space of exponentially increasing functions as $t \to \infty$,
- $F_\eta(\mathbb{R}, X) = \left\{u \in C^0(\mathbb{R}, X) \text{ with } \|u\|_{F_\eta} = \sup_{t\in\mathbb{R}} e^{\eta t}\|u(t)\|_X < \infty\right\}$, the space of functions which may grow exponentially at $-\infty$ and tend to 0 exponentially fast as $t \to \infty$.

Hypothesis 3.16 ([24])**.** *In* (3.34) *we assume that:*

- *$L : Z \to X$ is a linear operator,*
- *$R \in C^k(Z, Y)$ for some $k \geq 2$ and*

$$R(0) = 0, \quad DR(0) = 0.$$

In other words, we assume that $u = 0$ is a steady state of (3.34)*, and that R is nonlinear in u.*

Definition 3.17 ([24])**.** A solution of (3.34) is a function $u : I \to Z$ defined on an interval $I \subset \mathbb{R}$ such that

- $u : I \to Z$ is continuous,
- $u : I \to Z$ is continuously differentiable,
- equality (3.34) holds in X for all $t \in I$.

For the theorem to hold, we require the following two assumptions on the spectrum of L.

Hypothesis 3.18 (Spectral gap,[24])**.** *Consider the spectrum σ of L and write*

$$\sigma = \sigma_+ \cup \sigma_0 \cup \sigma_-,$$

where

$$\sigma_+ = \{\lambda \in \sigma, \mathrm{Re}\lambda > 0\}, \quad \sigma_0 = \{\lambda \in \sigma, \mathrm{Re}\lambda = 0\}, \quad \sigma_+ = \{\lambda \in \sigma, \mathrm{Re}\lambda > 0\}.$$

These subsets of σ are called unstable, center *and* stable *spectrum respectively. Assume that*

1. *there exists $\gamma > 0$ such that (see Figure 3.6)*

$$\inf_{\lambda\in\sigma_+} \mathrm{Re}\lambda > \gamma, \quad \sup_{\lambda\in\sigma_-} \mathrm{Re}\lambda < -\gamma,$$

2. *the center spectrum σ_0 consists of a finite number of eigenvalues of finite algebraic multiplicities.*

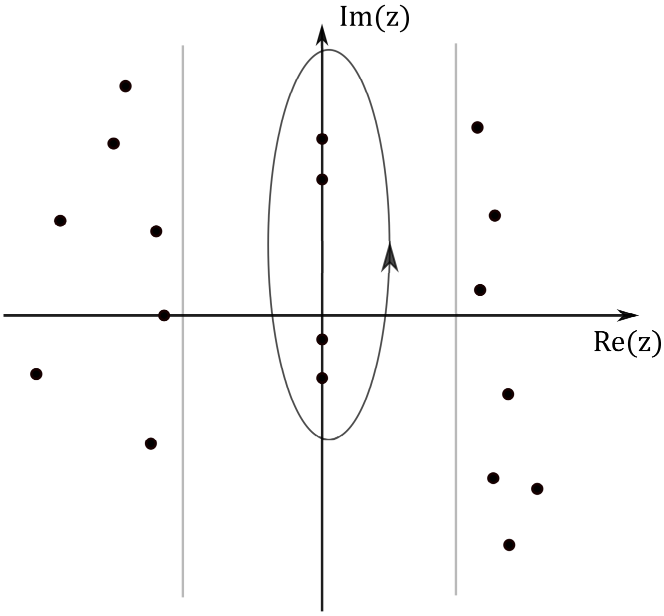


Figure 3.6: The requirement on the spectrum of L: on the imaginary axis Im$z = 0$ only a finite number of finite multiplicity eigenvalues is allowed. Also, the rest of the spectrum should be bounded away from the imaginary axis. This allows us to find a curve Γ that encircles the center spectrum which is needed in order to define the spectral projection P_0.

Given Hypothesis 3.18, we are able to find a simple, smooth curve surrounding σ_0 so that the Dunford integral $P_0 : X \to X$,

$$P_0 = \frac{1}{2\pi i}\int_\Gamma (z-L)^{-1}\,dz,$$

is a well-defined and bounded linear operator. In fact, P_0 is a projection onto the finite-dimensional subspace $\mathscr{E}_0$, spanned by the eigenfunctions of the eigenvalues of the center spectrum σ_0. Thus, P_0 has the following properties:

- $P_0^2 = P_0$,
- $P_0 L = L P_0$ for all $u \in Z$,
- range $P_0 = \mathscr{E}_0 \subset Z$, thus $P_0 \in \mathscr{L}(X, Z)$ as well.

The complementary projection, $P_h : X \to X$ given by $P_h = 1 - P_0$ has similar properties:

- $P_h^2 = P_h$,
- $P_h L u = L P_h u$ for all $u \in Z$.

Moreover, we have

$$\mathscr{E}_0 = \operatorname{range} P_0 = \ker P_h \subset Z, \quad X_h := \operatorname{range} P_h = \ker P_0 \subset X$$

and X can be decomposed as following:

$$X = \mathscr{E}_0 \oplus X_h.$$

Before stating the final hypothesis, set

$$Z_h = P_h Z \subset Z, \quad Y_h = P_h Y \subset Y,$$

and let L_0, L_h to be the restrictions of L to $\mathscr{E}_0$ and X_h respectively. This means that $\sigma(L_0) = \sigma_0$ and $\sigma(L_h) = \sigma_+ \cup \sigma_-$.

Hypothesis 3.19 (Linear equation, [24])**.** *For any $\eta \in [0,\gamma]$ and any $f \in C_\eta(\mathbb{R}, Y_h)$ the linear problem*

$$\frac{\mathrm{d}u_h}{\mathrm{d}t} = L_h u_h + f(t)$$

has a unique solution $u_h = K_h f \in C_\eta(\mathbb{R}, Z_h)$. Furthermore, $K_h \in \mathscr{L}\big(C_\eta(\mathbb{R}, Y_h), C_\eta(\mathbb{R}, Z_h)\big)$ and there exists a continuous $C : [0,\gamma] \to \mathbb{R}$ such that

$$\|K_h\|_{\mathscr{L}(C_\eta(\mathbb{R},Y_h),C_\eta(\mathbb{R},Z_h))} \le C(\eta).$$

It is no surprise that Hypothesis 3.19 will, in general, be much harder to check than Hypothesis 3.16 and Hypothesis 3.18. An alternative assumption, which is sufficient but not necessary for Hypothesis 3.19 to hold is given below.

Hypothesis 3.20 (Resolvent estimate, [24])**.** *Let X, Y, Z be* Hilbert spaces *and assume that there exist positive constants $\omega_0, c > 0$, such that for all $\omega \in \mathbb{R}$, with $|\omega| \ge \omega_0$ we have that $i\omega \in \rho(L)$ and*

$$\|(i\omega - L)^{-1}\|_{\mathscr{L}(X)} \le \frac{c}{|\omega|}.$$

3.3.2 Statement of the theorem

We are now ready to state the center manifold theorem.

Theorem 3.21 (Center manifold theorem, [24, Chapter 2, Theorem 2.9])**.** *Let Hypothesis 3.16, Hypothesis 3.18 and Hypothesis 3.19 hold. Then there exists a map $\Psi \in C^k(\mathscr{E}_0, Z_h)$ called the* reduction function*, with*

$$\Psi(0) = 0, \quad D\Psi(0) = 0,$$

and a neighbourhood U of 0 in Z such that the manifold

$$\mathscr{M}_0 = \{u_0 + \Psi(u_0), u_0 \in \mathscr{E}_0\} \subset Z$$

has the following properties:

1. *$\mathscr{M}_0$ is locally invariant, i.e if u is a solution of (3.34) satisfying $u(0) \in \mathscr{M}_0 \cap U$ and $u(t) \in U$ for all $t \in [0,T]$, then $u(t) \in \mathscr{M}_0$ for all $t \in [0,T]$.*

2. *$\mathscr{M}_0$ contains the set of bounded solutions of (3.34) staying in U for all $t \in \mathbb{R}$, i.e, if u is a solution of (3.34) satisfying $u(t) \in U$ for all $t \in \mathbb{R}$, then $u(0) \in \mathscr{M}_0$.*

A solution u of (3.34) on the center manifold $\mathscr{M}_0$ has the form $u(t) = u_0(t) + \Psi(u_0)$, where $u_0(t) \in \mathscr{E}_0$, from the previous theorem. Substituting this into (3.34), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_0 + \Psi(u_0)) = L(u_0 + \Psi(u_0)) + R(u_0 + \Psi(u_0)),$$

or, equivalently

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} + D\Psi(u_0)\frac{\mathrm{d}u_0}{\mathrm{d}t} = L_0 u_0 + L_h\Psi(u_0) + R(u_0 + \Psi(u_0)).$$

Applying the projection P_0 to both sides of the last equality, we see that u_0 satisfies the equation

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = L_0 u_0 + P_0 R(u_0 + \Psi(u_0)), \tag{3.35}$$

because $D\Psi(u_0) \in \mathscr{L}(\mathscr{E}_0, Z_h)$ and thus $D\Psi(u_0)\frac{\mathrm{d}u_0}{\mathrm{d}t}$ vanishes under P_0. Note that this a *finite-dimensional* problem. Thus, on the center manifold, the original infinite-dimensional problem reduces to a finite-dimensional one.

It is often the case that we have a parameter-dependent problem, that is, a problem of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Lu + R(u, \mu), \tag{3.36}$$

for some, constant in t, parameter $\mu \in \mathbb{R}^m$.

Hypothesis 3.22. *Assume that L and R in (3.36) have the following properties:*

1. $L \in \mathscr{L}(Z, X)$,
2. *for some* $k \geq 2$, $R \in C^k(Z \times \mathbb{R}^m, Y)$ *and*

$$R(0,0) = 0, \quad D_u R(0,0) = 0.$$

Hypothesis 3.22 implies that for $\mu = 0$, $u = 0$ is a steady state for (3.36).

Remark 3.23. The parameter μ appears in the nonlinear part R only, so the linear operator L is independent of μ. Moreover we assume that $\mu \in \mathbb{R}^m$ is a finite-dimensional quantity.

By appending the trivial equation $\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0$ to (3.36) and applying the center manifold theorem, we obtain the next result.

Theorem 3.24 (Parameter-dependent center manifolds, [24, Chapter 2, Theorem 3.3])**.** *Assume that Hypothesis 3.22, Hypothesis 3.18 and Hypothesis 3.19 hold. Then, there exists a map* $\Psi \in C^k(\mathscr{E}_0 \times \mathbb{R}^m, Z_h)$, *called* reduction function, *with*

$$\Psi(0,0) = 0, \quad D_u\Psi(0,0) = 0,$$

and a neighbourhood $U \times V$ *of* $Z \times \mathbb{R}^m$ *such that for* $\mu \in V$ *the manifold*

$$\mathscr{M}_0(\mu) = \{u_0 + \Psi(u_0, \mu), u_0 \in \mathscr{E}_0\}$$

has the following properties:

1. *$\mathcal{M}_0(\mu)$ is locally invariant, i.e, if u is a solution of* (3.36) *with $u(0) \in \mathcal{M}_0(\mu) \cap U$ and $u(t) \in U$ for all $t \in [0, T]$, then $u(t) \in \mathcal{M}_0(\mu)$ for all $t \in [0, t]$.*

2. *$\mathcal{M}_0(\mu)$ contains the set of bounded solutions of* (3.36) *staying in U for all $t \in \mathbb{R}$, that is, if u is a solution of* (3.36) *such that $u(t) \in U$ for all $t \in \mathbb{R}$, then $u(0) \in \mathcal{M}_0(\mu)$.*

In the parameter-dependent case, the reduced system, analogous to (3.35), is

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = L_0 u_0 + P_0 R\big(u_0 + \Psi(u_0, \mu), \mu\big) \tag{3.37}$$

Note that since $\Psi : \mathcal{E}_0 \times \mathbb{R}^n \to Z_h$ and $Z_h = P_h Z \subset P_h X = \operatorname{range} P_h = \ker P_0$ it follows that $Z_h \subset \ker P_0$ and, in particular,

$$P_0\big(\Psi(u_0, \mu)\big) = 0. \tag{3.38}$$

Remark 3.25. In the case that the geometric and algebraic multiplicities of the eigenvalues of σ_0 are equal, as will happen in our problem, L_0 will be the trivial operator and the reduced system will take the form

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = P_0 R\big(u_0 + \Psi(u_0, \mu), \mu\big). \tag{3.39}$$

Frequently, especially when L is a second order elliptic differential operator, it happens that the unstable spectrum σ_+ of L is empty. In this case, the center manifold will be *locally attracting*, and Theorem 3.24 can be strengthened as follows.

Theorem 3.26 (Empty unstable spectrum, [24, Chapter 2, Theorem 3.22])**.** *Assume that Hypothesis 3.22, Hypothesis 3.18 and Hypothesis 3.20 hold. In addition to the conclusions of Theorem 3.24, the local center manifold $\mathcal{M}_0(\mu)$ is locally attracting, in the sense that any solution of* (3.36) *that stays in U for all $t > 0$ tends exponentially towards a solution of* (3.36) *on the center manifold $\mathcal{M}_0(\mu)$. More precisely, if $u(0) \in U$ and the solution $u(t; u(0))$ of* (3.36) *satisfies $u(t; u(0)) \in U$ for all $t > 0$, there exists $\tilde{u} \in \mathcal{M}_0(\mu) \cap U$ and $c > 0$ such that*

$$u(t; u(0)) = u(t; \tilde{u}) + O(e^{-ct}) \quad \textit{as } t \to \infty.$$

3.3.3 Slowly varying parameters

Of particular interest in our study of bifurcation delay are systems of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Lu + R(u, \mu, \varepsilon), \tag{3.40a}$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \varepsilon, \tag{3.40b}$$

for small $0 < \varepsilon \ll 1$, where $u(t) \in Z$ and $\mu \in \mathbb{R}^m$ and the assumptions of Theorem 3.24 hold for the first equation. To be precise, Z is the domain of $L : Z \to X$, while

$R : X \times \mathbb{R}^m \times \mathbb{R} \to Y$ is a map such that

$$R(0,0,0) = 0, \quad D_{(u,\mu)} R(0,0,0) = 0.$$

This system may be seen as a variation of (3.36) where the parameter μ is a slowly evolving variable. We want to investigate how center manifold reduction applies to (3.40). To this end, we apply the parametric version (Theorem 3.24) with ε regarded as the parameter and (u,μ) as the state variables. The linear operator $\widetilde{L} : Z \times \mathbb{R}^m \to X \times \mathbb{R}^m$ is

$$\widetilde{L} = \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix},$$

while the nonlinear part $\widetilde{R} : Z \times \mathbb{R}^m \times \mathbb{R} \to Y \times \mathbb{R}^m$ is given by

$$\widetilde{R}(u,\mu,\varepsilon) = \begin{pmatrix} R(u,\mu,\varepsilon) \\ \varepsilon \end{pmatrix}.$$

It is immediate to check that

$$\sigma_{\widetilde{L}} = \sigma(L) \cup \{0\},$$

while

$$\widetilde{\mathscr{E}}_0 = (\mathscr{E}_0 \times \{0\}) \cup (\{0\} \times \mathbb{R}^m) \subset Z \times \mathbb{R}^m. \tag{3.41}$$

As L and R satisfy the Hypothesis required by Theorem 3.24, it follows that these are satisfied for $\widetilde{L}$ and $\widetilde{R}$ as well. Also, it is possible to modify the curve Γ (see Hypothesis 3.18) to include the eigenvalue $\lambda = 0$ if required.

The spectral projection $\widetilde{P}_0 \in \mathscr{L}(X \times \mathbb{R}^m)$ is given by

$$\widetilde{P}_0 = \begin{pmatrix} P_0 & 0 \\ 0 & P_\mu \end{pmatrix}.$$

The operator P_μ in the second component of the last equality is simply the identity, as follows from a straightforward calculation:

$$P_\mu = \frac{1}{2\pi i} \int_\Gamma (z\mathbb{I})^{-1} dz = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z} \mathbb{I}\, dz = \mathbb{I}.$$

Thus, we have

$$\widetilde{P}_0 = \begin{pmatrix} P_0 & 0 \\ 0 & 1 \end{pmatrix}$$

and consequently

$$\widetilde{X}_h = X_h \times \{0\}, \quad \widetilde{Z}_h = Z_h \times \{0\}.$$

This means that the reduction function $\widetilde{\Psi} : \widetilde{\mathscr{E}}_0 \times \mathbb{R} \to \widetilde{Z}_h$ has the form

$$\widetilde{\Psi}(u,\mu,\varepsilon) = \begin{pmatrix} \Psi(u,\mu,\varepsilon) \\ 0 \end{pmatrix},$$

where, $\Psi : \mathscr{E}_0 \times \mathbb{R}^m \times \mathbb{R} \to Z_h$ giving rise to a center manifold $\widetilde{\mathscr{M}_0}(\varepsilon) \subset Z \times \mathbb{R}^m$ that can be written, using (3.41) as

$$\widetilde{\mathcal{M}_0}(\varepsilon) = \mathcal{M}_0(\varepsilon) \times \mathbb{R}^m = \{u_0 + \Psi(u_0, \mu, \varepsilon), u_0 \in \mathcal{E}_0, \mu \in \mathbb{R}^m\} \times \mathbb{R}^m.$$

Regarding the reduced system, the form of the projection $\widetilde{P}_0$, along with the reduced equation (3.37), imply that the system is

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = L_0 u_0 + P_0 R\left(u_0 + \Psi(u_0, \mu, \varepsilon), \mu, \varepsilon\right), \tag{3.42a}$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \varepsilon, \tag{3.42b}$$

and the following theorem is proved.

Theorem 3.27. *The center manifold* $\widetilde{\mathcal{M}_0}(\varepsilon)$ *of system* (3.40) *can be written as*

$$\widetilde{\mathcal{M}_0}(\varepsilon) = \mathcal{M}_0(\varepsilon) \times \mathbb{R}^m = \{u_0 + \Psi(u_0, \mu, \varepsilon),\ (u_0, \mu) \in \mathcal{E}_0 \times \mathbb{R}^m\} \times \mathbb{R}^m,$$

where the reduction function $\Psi : \mathcal{E}_0 \times \mathbb{R}^m \times \mathbb{R} \to Z_h$ *is such that*

$$\Psi(0,0,0) = 0, \quad D\Psi_{(u,\mu)}(0,0,0) = 0,$$

and the reduced system is given by (3.42)*. This reduction is local in the sense of Theorem 3.24, and valid in a neighbourhood* $U \times V$ *of* 0 *in* $X \times \mathbb{R}^m$ *for* $\varepsilon \in [0, \varepsilon_0]$ *with* $\varepsilon_0 > 0$ *being a constant. Moreover, if the unstable spectrum* σ_+ *of* L *is empty,* $\widetilde{\mathcal{M}_0}(\varepsilon)$ *and* $\mathcal{M}_0(\varepsilon)$ *are locally attracting, in the sense of Theorem 3.26.*

Summarizing this subsection, whenever we are dealing with systems like (3.40), we are concerned with the first component only, as the reduction process leaves the second equation unchanged. This is to be expected, as the linear operator of the second part is the trivial identically zero function, with a single eigenvalue, $\lambda = 0$ and the eigenspace being simply $\mathbb{R}^m$, so in a sense the equation is already simplified as much as possible.

We also see why the slow variable μ has to be finite-dimensional: if, for example, we allow $\mu \in H$, with H an infinite-dimensional space, then eigenspace of the zero eigenvalue would be H, which violates Hypothesis 3.18 where we require that all the center eigenvalues have finite multiplicities.

Remark 3.28. The argument in this subsection can be adopted to work with a system of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Lu + R(u, \mu, \varepsilon),$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \varepsilon G(u, \mu, \varepsilon),$$

for functions $G : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ sufficiently smooth.

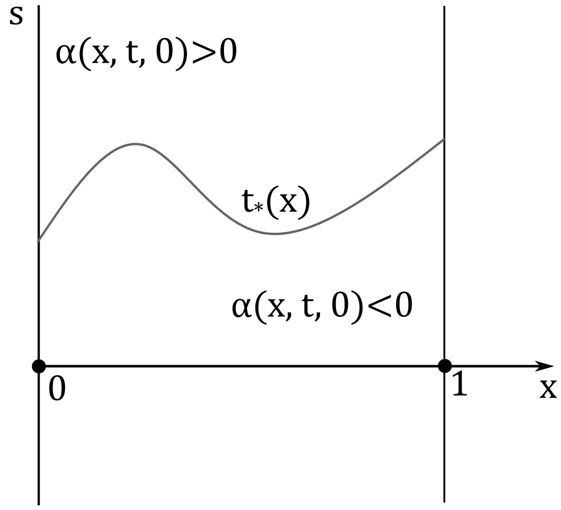


Figure 3.7: Sign of the turning point curve $a(x,t,0)$.

3.4 Bifurcation delay in linear reaction-diffusion equations

Recall that we are working with singularly perturbed linear reaction diffusion equations of the form

$$\varepsilon u_t = \delta u_{xx} + a(x,t,\varepsilon)u, \tag{3.43}$$

with $x \in [0,1]$ and $t \in [0, T_{\max}]$ with homogeneous Neumann boundary conditions

$$u_x(0,t) = u_x(1,t) = 0,$$

and initial condition $u(x,0)$. By renaming time t to s and introducing it as a state variable by appending the equation $\frac{\mathrm{d}s}{\mathrm{d}t} = 1$, so that the system becomes autonomous, and upon rescaling the new time t by $t \mapsto \frac{1}{\varepsilon}t$, we obtain the equivalent PDE-ODE system

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \delta u_{xx} + a(x,s,\varepsilon)u, \tag{3.44a}$$
$$\frac{\mathrm{d}s}{\mathrm{d}t} = \varepsilon, \tag{3.44b}$$

for $0 < \varepsilon \ll 1$ and some constant $\delta > 0$. We assume that the function $a : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is smooth (C^k for some $k \geq 2$) and has a $t_* \in C^1([0,1])$ curve of turning points $t_* = t_*(x)$, $t_*(x) > 0$, for $\varepsilon = 0$, see Figure 3.7. In other words, $a(x, t_*(x), 0) = 0$ and

$$\begin{aligned} a(x,s,0) &> 0 \quad \text{for } s > t_*(x), \\ a(x,s,0) &< 0 \quad \text{for } s < t_*(x). \end{aligned} \tag{3.45}$$

Remark 3.29. The requirement that $t_*(x) > 0$ is there so that the turning points occur for positive time t in original, non-autonomous equation (3.43).

Consider the Hilbert spaces $X := L^2(0,1)$ and $Z := \{u \in H^2(0,1) | u'(0) = u'(1) = 0\}$.

Note that $Z \hookrightarrow X$ with dense and continuous embedding. Define the linear operator $L_{s,\varepsilon} : Z \to X$ by

$$L_{s,\varepsilon} u := \delta u_{xx} + a(x,s,\varepsilon)u. \tag{3.46}$$

With this in mind, we may view system (3.44) as an infinite dimensional dynamical system on $X \times \mathbb{R}$, regarding ε as a parameter:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = L_{s,\varepsilon} u, \tag{3.47a}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \varepsilon. \tag{3.47b}$$

We wish to analyze this equation by making use of the center manifold theorem. However, the linear operator $L_{s,\varepsilon}$ depends on the state variable s and the parameter ε, so the theorem cannot be applied immediately. Instead, we first need to identify any bifurcation points, and take the linear operator to be L at that point.

Definition 3.30. A *bifurcation point* for equation (3.47) is a pair of real numbers (s_*, ε_*) such that the operator L_{s_*,ε_*} has non-empty center spectrum σ_0.

Since we are interested in what happens in the singular limit $\varepsilon \to 0$, we always will look for bifurcation points of the form $(s_*, 0)$.

3.4.1 The simplest case

In this subsection we consider system (3.44) with

$$a(x,s,\varepsilon) = s - 1, \tag{3.48}$$

so that the loss of stability occurs at $s = 1$. We could take directly $a(x,s,\varepsilon) = s$ but then we would have to start with initial value $s < 0$. Arguably, this is the simplest possible choice for a, it is instructing to see how the application of the center manifold theorem works. Thus, the system we are working with is

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \delta u_{xx} + (s-1)u,$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \varepsilon,$$

It is somewhat more convenient to translate s by $s \mapsto s + 1$, so that we have

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \delta u_{xx} + su, \tag{3.50a}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \varepsilon, \tag{3.50b}$$

Next, we set $\varepsilon = 0$ and describe the spectrum of $L_{s,0}$.

Proposition 3.31. *With $a(x,s,\varepsilon) = s$, the spectrum of the operator $L_{s,0}$ consists of the simple (geometrically and algebraically) eigenvalues*

$$\lambda_k = s - \delta k^2 \pi^2, \quad k = 0, 1, 2, \ldots$$

We identify the bifurcation points at $s = \delta k^2\pi^2$, $k = 0,1,2,\dots$. Of these, we are concerned with the one occurring for $k = 0$, at $s = 0$, as this is where the solution $u = 0$ of (3.50) loses stability, as the eigenvalue $\lambda_0(s) = s$ crosses from the left half-plane $\{\mathrm{Re}\, z < 0\}$ to the right half-plane $\{\mathrm{Re}\, z > 0\}$. The corresponding eigenfunction, $v_0 : [0,1] \to \mathbb{R}$ is simply the constant function

$$v_0(x) = 1$$

and the corresponding eigenspace is

$$\mathscr{E}_0 = \{A v_0(x) | A \in \mathbb{R}\} \subset Z.$$

To apply the center manifold theorem, we rewrite (3.50) as

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= L_{0,0}u + (L_{s,\varepsilon} - L_{0,0})u,\\ \frac{\mathrm{d}s}{\mathrm{d}t} &= \varepsilon,\end{aligned}$$

and take $R : Z \to X$,

$$R(u,s,\varepsilon) = (L_{s,\varepsilon} - L_{0,0})u = su$$

while considering ε as a parameter. Now we can calculate the reduced equation on the center manifold using Theorem 3.27. Take $u_0(t) := A(t)v_0(x) \in \mathscr{E}_0$. Then,

$$\begin{aligned}P_0 R(u_0 + \Psi(u_0,s,\varepsilon)) &= P_0(su_0 + s\Psi(u_0,s,\varepsilon))\\ &= sP_0u_0 + sP_0\Psi(u_0,s,\varepsilon)\\ &= su_0,\end{aligned}$$

using the observation (3.38). From (3.39), we obtain

$$\frac{\mathrm{d}A}{\mathrm{d}t} = sA(t),$$

which along with $\frac{\mathrm{d}s}{\mathrm{d}t} = \varepsilon$ and after undoing the translation of s, gives the reduced system

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= (s-1)A(t),\\ \frac{\mathrm{d}s}{\mathrm{d}t} &= \varepsilon.\end{aligned}$$

Because all the other eigenvalues of $L_{0,0}$ apart from λ_0 have negative real parts, the center manifold is *attracting*. This suffices to deduce the presence of a bifurcation delay in (3.50), as the above ODE exhibits bifurcation delay as we saw in section 3.1, equation (3.3).

Remark 3.32. In general, the center manifold reduction would be valid in a neighbourhood $(u,s) \in U$ of the origin in $Z \times \mathbb{R}$, for $\varepsilon \in [0,\varepsilon_0]$. However, in this case, the reduction is *global*, that is, $U = Z \times \mathbb{R}$. The global nature of the reduction follows by noting that if we consider the neighbourhood $(u,s) \in U_r = B_r(Z) \times [-r,r]$ then

$$\sup_{u,s\in U_r} \|R(u,s,\varepsilon)\|_X = \mathcal{O}\left(r^2\right) \quad \text{as } r \to \infty.$$

and

$$\sup_{(u,s)\in U_r} \|D_u R(u,s,\varepsilon)\| = \sup_{(u,s)\in U_r} \|D_s R(u,s,\varepsilon)\| = \mathcal{O}(r) \quad \text{as } r \to \infty.$$

The norm in the second equality is the operator norm. See [24, Appendix B.1]. This allows us to calculate the exit time of the PDE as the exit time of the reduced system.

3.4.2 Space independent turning point curve

Let us continue by considering the case where $a(x,s,\varepsilon)$ is of the form

$$a(x,s,\varepsilon) = a(s,\varepsilon)$$

with

$$\begin{aligned} a(s,0) &> 0, \quad \text{for } s > s_*, \\ a(s,0) &< 0, \quad \text{for } s < s_*, \\ a(s,0) &= 0, \quad \text{for } s = s_* \end{aligned}$$

for some $s_* > 0$. The system, this time, is

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \delta u_{xx} + a(s,\varepsilon)u, \tag{3.53a}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \varepsilon. \tag{3.53b}$$

Without loss of generality, we may assume that $s_* = 0$. The treatment of this case is similar to the one in subsection 3.4.1. Like previously, we set $\varepsilon = 0$, and look for bifurcation points of the first equation.

Proposition 3.33. *The spectrum of the linear operator*

$$L_{s,0}u = \delta u_{xx} - a(s,0)u$$

consists of the simple eigenvalues

$$\lambda_k = a(s,0) - \delta k^2 \pi^2, \quad k = 0,1,2,\ldots$$

Again, we are interested in the change of sign of the largest eigenvalue $\lambda_0(s) = a(s,0)$ at $s_* = 0$. Write

$$\frac{\mathrm{d}u}{\mathrm{d}t} = L_{0,0}u + (L_{s,\varepsilon} - L_{0,0})u,$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \varepsilon,$$

and apply Theorem 3.27 with $R(u,s,\varepsilon) = (L_{s,\varepsilon} - L_{0,0})u = a(s,\varepsilon)u$. We find that the reduced system is

$$\begin{aligned}\frac{\mathrm{d}u_0}{\mathrm{d}t} &= P_0 R\left(u_0 + \Psi(u_0,s,\varepsilon),s,\varepsilon\right)\\ &= P_0\left(a(s,\varepsilon)u_0 + a(s,\varepsilon)\Psi(u_0,s,\varepsilon)\right)\\ &= a(s,\varepsilon)P_0 u_0 + a(s,\varepsilon)\Psi(u_0,s,\varepsilon)\\ &= a(s,\varepsilon)u_0.\end{aligned}$$

Thus, the reduced system for this case is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = a(s,\varepsilon)A, \tag{3.54a}$$
$$\frac{\mathrm{d}s}{\mathrm{d}t} = \varepsilon. \tag{3.54b}$$

Once more, the center manifold is attracting since all the eigenvalues of $L_{0,0}$ but the zero one are negative. Since the reduced system has bifurcation delay (it is of type (3.3)), the same is true for the full system.

Remark 3.34. If $a(s,0)$ changes sign at some point $s_* \neq 0$, then the reduced system is still given by the same expression.

Remark 3.35. The center manifold reduction is global in u but not in s in this case, that is, we may take $U \times V$ in Theorem 3.27 to be $Z \times [-r,r]$ for $s \in [-r,r]$ with arbitrary, but finite $r > 0$. Of course, we must also have $\varepsilon \in [0,\varepsilon_0]$, with $\varepsilon_0 > 0$ constant. The restriction on s is there to guarantee that

$$\|R(u,s,\varepsilon)\|_X \leq C\|u\|_Z,$$

with

$$C = \max_{|s|\leq r, |\varepsilon|\leq\varepsilon_0} a(s,\varepsilon).$$

We assume that $a(s,t)$ is at least twice differentiable, so C is finite for finite r. If $a(s,\varepsilon)$ grows linearly in s as $s \to \pm\infty$ then the reduction is global in s as well. Despite the restriction on s, this suffices to identify the exit time of (3.53) as the exit time of (3.54).

3.4.3 Space dependent turning point curve

In this subsection we treat a more general form of $a(x,s,\varepsilon)$, namely,

$$a(x,s,\varepsilon) = s - a(x,\varepsilon),$$

with $a(x,\varepsilon) \in C^1$. For $\varepsilon = 0$, we have the smooth curve of turning points $a(x,0)$. As per (3.45) we assume that $a(x,0) > 0$, although this is not essential, as already explained in the beginning of this section. The system this time is

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= u_{xx} + (s - a(x,\varepsilon))u,\\ \frac{\mathrm{d}s}{\mathrm{d}t} &= \varepsilon,\end{aligned}$$

with the differential operator $L_{s,\varepsilon}$ given by

$$L_{s,\varepsilon}u = u_{xx} + (s - a(x,\varepsilon))u. \tag{3.56}$$

We set $\varepsilon = 0$ and look to describe the spectrum of $L_{s,0}$. This is the difficult part of the analysis, as it is not possible to give explicit expressions for the eigenvalues. However, $L_{s,0}$ is a particular case of a Sturm-Liousville operator and is well-studied [49]. The following proposition contains what we need to proceed.

Proposition 3.36. *Consider the operator $T : H^2(0,1) \to L^2(0,1)$,*

$$Tu = u_{xx} + q(x)u,$$

with Dirichlet or Neumann boundary conditions and $q \in L^1(0,1)$. The following assertions hold for the spectrum of A:

1. *It consists of discrete, real, simple eigenvalues λ_k, $k = 0,1,2,\dots$ that can be ordered so that*
$$\dots < \lambda_2 < \lambda_1 < \lambda_0 < +\infty, \quad \lambda_k \to -\infty.$$
2. *The corresponding eigenfunction $v_k : [0,1] \to \mathbb{R}$ has exactly k zeros. In particular, $v_0(x)$ can be taken strictly positive.*
3. *Each eigenvalue λ_k depends continuously on q.*
4. *Each eigenvalue $\lambda_k(q)$ viewed as a function of q, is increasing with q, that is, if $Q \geq q$ on $[0,1]$ then $\lambda_k(Q) \geq \lambda_k(q)$ and if $Q > q$ on a subset of $[0,1]$ of positive measure then $\lambda_k(Q) > \lambda_k(q)$.*

Proof. See [49, Chapter 4] and more specifically Theorems 4.6.2, 4.9.1. □

Returning to (3.56), for $L_{s,0}$, observe that if $s > \max_{x\in[0,1]} a(x,0)$ Proposition 3.36 implies that $\lambda_0(s) > 0$, while if $s < \min_{x\in[0,1]} a(x,0)$ $\lambda_0(s) < 0$. What is more, since $s - a(x)$ is increasing with s, also from Proposition 3.36, it follows that $\lambda_0(s)$ is continuous and increasing. Thus there exists a unique s_* with

$$\min_{x\in[0,1]} a(x,0) \leq s_* \leq \max_{x\in[0,1]} a(x,0),$$

such that $\lambda(s_*) = 0$ and $\lambda(s) < 0, \lambda(s) > 0$ for $s < s_*$ and $s > s_*$ respectively. In the following, without loss of generality, we assume that $s_* = 0$ by replacing s with $s + s_*$ if required.

With this in mind, at $s = 0$ the first eigenvalue crosses from the negative to the positive part of the real axis, causing an exchange of stability. To analyze the behaviour of the system at this bifurcation point, we employ the center manifold reduction in the form of Theorem 3.27 by writing the system as

$$\frac{\mathrm{d}u}{\mathrm{d}t} = L_{0,0}u + (L_{s,\varepsilon} - L_{0,0})u, \tag{3.57a}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \varepsilon. \tag{3.57b}$$

In other words, we take $L = L_{0,0}$ and $R(u,s,\varepsilon) = (L_{s,\varepsilon} - L_{0,0})u = (s - a(x,\varepsilon) + a(x,0))u$. As previously,

$$\mathscr{E}_0 = \operatorname{span}\{v_0(x)\} \subset Z,$$

so we take a solution $u_0(t) = A(t)v_0(x) \in \mathscr{E}_0$ on the center manifold and calculate

$$\begin{aligned}\frac{\mathrm{d}u_0}{\mathrm{d}t} &= P_0 R(u_0 + \Psi(u_0,s,\varepsilon),s,\varepsilon)\\ &= P_0(s - a(x,\varepsilon) + a(x,0))(u_0 + \Psi(u_0,s,\varepsilon))\\ &= P_0(s - a(x,\varepsilon) + a(x,0))u_0\\ &= (s - a_1(\varepsilon) + a_1(0))Av_0,\end{aligned}$$

where the newly introduced function $a_1(\varepsilon)$ is defined by

$$P_0(a(x,\varepsilon)v_0(x)) = a_1(\varepsilon)v_0(x).$$

We conclude that the reduced system is

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= (s - a_1(\varepsilon) + a_1(0))A,\\ \frac{\mathrm{d}s}{\mathrm{d}t} &= \varepsilon.\end{aligned}$$

This slow-fast ODE is of the type exhibiting bifurcation delay (3.3). Like in the previous cases, $\lambda_k(0) < 0$ for all $k = 1,2,\ldots$, therefore the center manifold is attracting.

Remark 3.37. By undoing the shift of s, the reduced system changes to

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= (s - s_* - a_1(\varepsilon) + a_1(0))A,\\ \frac{\mathrm{d}s}{\mathrm{d}t} &= \varepsilon.\end{aligned}$$

Remark 3.38. The region of validity for the reduction is the same as in subsection 3.4.2, $(u,s,\varepsilon) = Z \times [-r,r] \times [0,\varepsilon_0]$ for arbitrarily large but finite $r > 0$. Still, this is enough to allow the calculation of the exit time of the full system (3.57) from the reduced system.

3.4.4 General space dependent turning point curve

The full general case with $a(x,s,\varepsilon)$ as in (3.45), is similar to the previous one. The difference is that $\lambda_0(s)$ may change sign multiple times, not just once. In this case we apply the center manifold theorem to the *least* s such that $\lambda_0(s) = 0$.

Therefore, we let

$$s_* = \min\{s \in \mathbb{R} \text{ such that } \lambda_0(s) = 0\}.$$

This is well-defined, since from Proposition 3.36, the set of zeros of $\lambda_0(s)$ is a closed set ($\lambda_0(s)$ is continuous), that is bounded below ($\lambda_0(s) < 0$ for $s < \min_x t_*(x)$) and above ($\lambda_0(s) > 0$ for $s > \max_x t_*(x)$). For convenience, we shift s_* to the origin by the translation $s \mapsto s + s_*$. Similarly to the previous special cases, we have $\lambda_0(0) = 0$ and $\lambda_k(0) < 0$ for $k \geq 1$. Additionally, the same argument shows that

$$\min_{x\in[0,1]} t_*(x) \leq s_* \leq \max_{x\in[0,1]} t_*(x). \tag{3.58}$$

Let

$$R(u,s,\varepsilon) = (L_{s,\varepsilon} - L_{0,0})u = (a(x,s,\varepsilon) - a(x,0,0))\,u.$$

The variant of the center manifold reduction given in Theorem 3.27 applied to system

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= u_{xx} + a(x,s,\varepsilon)u = L_{0,0}u + (L_{s,\varepsilon} - L_{0,0})u = L_{0,0}u + R(u,s,\varepsilon),\\ \frac{\mathrm{d}s}{\mathrm{d}t} &= \varepsilon,\end{aligned}$$

at the point $(u,s) = (0,0)$, gives

$$\begin{aligned}\frac{\mathrm{d}u_0}{\mathrm{d}t} &= P_0R\,(u_0 + \Psi(u_0,s,\varepsilon),s,\varepsilon)\\ &= P_0(a(x,s,\varepsilon)\,u_0) - P_0(a(0,x,0)\,u_0)\\ &= a_1(s,\varepsilon)\,u_0 - a_1(0,0)\,u_0.\end{aligned}$$

Upon substituting $u_0 = A(t)\,v_0(x)$, one finds

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= (a_1(s,\varepsilon) - a_1(0,0))\,A,\\ \frac{\mathrm{d}s}{\mathrm{d}t} &= \varepsilon.\end{aligned}$$

The new function $a_1(s,\varepsilon)$, is defined by the relation

$$P_0\,(a(s,x,\varepsilon)\,v_0(x)) = a_1(s,\varepsilon)\,v_0(x). \tag{3.59}$$

Undoing the translation of s,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = (a_1(s,\varepsilon) - a_1(s_*,0))\,A, \tag{3.60a}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \varepsilon. \tag{3.60b}$$

This system, certainly looks like (3.3). For $\varepsilon = 0$, we have that the term in front of A vanishes at $s = s_*$. However, the change of sign behaviour around s_* is not clear. I suspect one can prove this by looking at (3.59) and taking the following points into account:

- The function $a(x,s_*,0)$ takes both positive and negative values as seen from (3.58).
- For $s > s_*$, the set $\{x \in [0,1] : a(x,s,0) > 0\}$ is strictly larger in measure than $\{x \in [0,1] : a(x,s_*,0) > 0\}$, and the other way around for $s < s_*$.
- The two points above combined with the fact that the eigenfunction $v_0(x)$

is strictly positive (remember Proposition 3.36), should suffice to show that $a_1(s,0) > a_1(s_*,0)$ for $s > s_*$ and $a_1(s,0) < a_1(s_*,0)$ for $s < s_*$.

Remark 3.39. The exit time of the full problem may be derived from (3.60), as in the previous subsections.

3.4.5 Concluding remarks

A benefit of the center manifold theory approach used in this section, is that the treatment of the nonlinear problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u_{xx} + a(x,s,\varepsilon)u + f(u,x,s,\varepsilon), \tag{3.61a}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \varepsilon, \tag{3.61b}$$

is almost identical to the linear one. Here, we assume that $f : Z \times [0,1] \times \mathbb{R} \times \mathbb{R} \to H^1(0,1)$ is nonlinear in u, that is,

$$D_u f(0,x,s,\varepsilon) = 0.$$

Due to the map f being function-valued, this allows for non-linearities such as

$$f(u,x,s,\varepsilon) = u^d, \quad f(u,x,s,\varepsilon) = \left(\frac{\partial u}{\partial x}\right)^2, \quad f(u,x,s,\varepsilon) = \int_0^1 u^2(x,t)\,dx, \quad \ldots$$

Due to technicalities related to the center manifold theorem, we assume that the values of f have additional regularity ($H^1(0,1)$ instead of $L^2(0,1)$). We may then proceed as in subsection 3.4.4 and apply center manifold reduction with

$$Lu = L_{0,0}u, \quad R(u,s,\varepsilon) = \left(L_{s,\varepsilon} - L_{0,0}\right)u + f(u,x,s,\varepsilon).$$

The reduced system in this nonlinear setting will take the form

$$\frac{\mathrm{d}A}{\mathrm{d}t} = (a_1(s,\varepsilon) - a_1(s_*,0))\,A + f_1(A,s,\varepsilon),$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \varepsilon.$$

where $f_1 = O(A^2)$ as $A \to 0$ is nonlinear in A. The rest of the symbols appearing have the meaning of subsection 3.4.4. In contrast to that subsection though, we lose the global validity in u of the reduction, and one can show that the bifurcation delay exists, but not estimate it.

Compared to the method of upper and lower solutions employed in [15], it is evident that the approach presented here can lead to much simpler arguments. On the other hand, is it not clear to apply the center manifold theory for $\alpha > 0$ in (3.6). This is because by taking $\varepsilon = 0$ in

$$L_{s,\varepsilon}u = \delta\varepsilon^{\alpha}u_{xx} + a(x,s,\varepsilon)u,$$

we find ourselves in a degenerate situation, in which setting $\varepsilon = 0$ gives a trivial operator. What is more, for the inhomogeneous exit time case $\alpha > 2$ (see second part of Theorem 3.2), we cannot hope hope to see this situation by reducing to an ODE, as an ODE naturally has only a single exit time. Overcoming these difficulties is a possible direction of future research.

For the $\alpha = 0$ case at least, the present analysis offers certain improvements over the corresponding result of [15], namely by lifting the requirement that the initial data $u(0)$ is strictly positive and allowing a wider class of non-linearities.

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